

# SYMMETRIES OF SUB-RIEMANNIAN SURFACES

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**ABSTRACT.** We obtain some results on symmetries of sub-Riemannian surfaces. In case of contact sub-Riemannian surface we base on invariants found by Hughen [12]. Using these invariants, we find conditions under which a sub-Riemannian surface does not admit symmetries. If a surface admits symmetries, we show how invariants help to find them. It is worth noting, that the obtained conditions can be explicitly checked for a given contact sub-Riemannian surface. Also, we consider sub-Riemannian surfaces which are not contact and find their invariants along the surface where the distribution fails to be contact.

## INTRODUCTION

A sub-Riemannian manifold is a  $k$ -dimensional distribution endowed by a metric tensor on an  $n$ -dimensional manifold. At present sub-Riemannian geometry is intensively studied, this is motivated by applications in various fields of science (see, e.g. the book [14], where many applications of sub-Riemannian geometry are presented; also, for interesting examples, we refer the reader to [3], [16], [18], where applications to mechanics, thermodynamics, and biology are given). At the same time, various aspects of the theory of symmetries of sub-Riemannian manifolds are widely investigated because symmetries are always of great importance for applications [4], [15]. Many papers are devoted to the theory of homogeneous (in part, symmetric) sub-Riemannian manifolds (see e.g. [7], [8], [12], [21]). The main investigation tool in these papers is the Lie algebras theory as is usual when we study homogeneous spaces.

In the present paper we study symmetries of sub-Riemannian surfaces, i.e. of sub-Riemannian manifolds with  $k = 2$  and  $n = 3$ . Our main goal is to give a practical tool (or an algorithmic procedure) for investigation of symmetries of a sub-Riemannian surface. The paper is organized as follows. In the first section we give in details construction of invariants of a contact sub-Riemannian surface using the Cartan reduction procedure (here we follow [12]) and show how to calculate them. In the second section we demonstrate how to apply invariants to finding symmetries of a contact sub-Riemannian surface. Finally, in the third section we consider a sub-Riemannian surface without assumption that it is contact and find invariants along the “singular surface”, where the distribution fails to be contact.

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## 1. CONTACT SUB-RIEMANNIAN SURFACES

Let  $M$  be an  $n$ -dimensional manifold and  $\Delta$  be a  $k$ -dimensional distribution on  $M$  endowed by a metric tensor field

$$(1) \quad \forall p \in M, \quad \langle \cdot, \cdot \rangle_p : \Delta_p \times \Delta_p \rightarrow \mathbb{R}.$$

Then  $(M, \Delta, \langle \cdot, \cdot \rangle)$  is called a *sub-Riemannian manifold* [14].

In the present paper we consider a *sub-Riemannian surface*  $\mathcal{S} = (M, \Delta, \langle \cdot, \cdot \rangle)$ , i.e. a two-dimensional distribution  $\Delta$  on a three-dimensional manifold  $M$ , where  $\Delta$  is endowed by a metric tensor field  $\langle \cdot, \cdot \rangle$ . In addition, we assume that *the distribution  $\Delta$  and the manifold  $M$  are oriented*. Note that we do not suppose that any metric on  $M$  is given.

Throughout the paper we will denote the Lie algebra of vector fields on a manifold  $N$  by  $\mathfrak{X}(N)$ , and the space of covector fields by  $\mathfrak{X}(N)^*$ . Also the space of  $r$ -forms on  $N$  will be denoted by  $\Lambda^r(N)$ .

1.1.  $G$ -structure associated with a sub-Riemannian surface.

1.1.1. *Elements of theory of  $G$ -structures.* Recall notions and results of the theory of  $G$ -structures we use in the present paper (for the details we refer the reader to [14] and [13]).

Tautological forms, pseudoconnection form, and structure equations. Let  $M$  be a smooth  $n$ -dimensional manifold, and  $\pi : B(M) \rightarrow M$  be the coframe bundle of  $M$ .

On  $B(M)$  the *tautological forms*  $\theta^a \in \Omega^1(B(M))$  are defined as follows [13]. For a point  $\xi \in B(M)$  ( $\xi = \{\xi^a\}_{a=\overline{1,n}}$  is a coframe of  $T_p M$ , where  $p = \pi(\xi)$ ), we set

$$(2) \quad \theta_\xi^a : T_\xi(B(M)) \rightarrow \mathbb{R}, \quad \theta_\xi^a(X) = \xi^a(d\pi(X)).$$

Now, on a neighborhood  $U$  of a point  $p \in M$ , take a coframe field  $\eta = \{\eta^a\}$ . This gives a trivialization  $\alpha : \pi^{-1}(U) \rightarrow U \times GL(n)$ : to a coframe  $\xi$  at  $p \in U$  we assign  $(p, g) \in U \times GL(n)$  such that  $\xi^a = \tilde{g}_b^a \eta_p^b$ , where  $||\tilde{g}_b^a|| = g^{-1}$ .

For a coframe field  $\eta$  on  $U$  let us consider the pullback 1-forms  $\bar{\eta}^a = d\pi^* \eta^a$  on  $U \times GL(n) \cong \pi^{-1}(U) \subset B(M)$ . Then

$$(3) \quad \theta_{(p,g)}^a = \tilde{g}_b^a \bar{\eta}_{(p,g)}^b = \tilde{g}_b^a d\pi^* \eta_p^b.$$

A  $G$ -structure  $P \rightarrow M$  is a principal subbundle of  $\pi : B(M) \rightarrow M$  with structure group  $G \subset GL(n)$ . The tautological forms on  $P$  are the restrictions of  $\theta^a$  to  $P$  and will be denoted by the same letters.

Let us denote by  $\mathfrak{g}$  the Lie algebra of the Lie group  $G$ . A *pseudoconnection form*  $\omega$  on a  $G$ -structure  $\pi : P \rightarrow M$  is a  $\mathfrak{g}$ -valued 1-form on  $P$  such that  $\omega(\sigma(a)) = a$ , where  $\sigma(a)$  is the fundamental vector field ([13], Ch. I, Sec. 5) on  $P$  corresponding to  $a \in \mathfrak{a}$ .

Given a pseudoconnection form  $\omega$ , we have *structure equations* on  $P$ :

$$(4) \quad d\theta^a = \omega_b^a \wedge \theta^b + T_{bc}^a \theta^b \wedge \theta^c$$

where the functions  $T_{bc}^a : P \rightarrow \mathbb{R}$  uniquely determined by equations (4) are called *torsion functions*, and the map  $T : P \rightarrow \Lambda^2 \mathbb{R}^n \otimes \mathbb{R}^n$ ,  $\xi \rightarrow \{T_{bc}^a(\xi)\}$ , is called the *torsion* of the pseudoconnection  $\omega_b^a$ .

**Structure function.** Let us find how the torsion changes under change of the pseudoconnection. If  $\omega_b^a, \hat{\omega}_b^a$  are pseudoconnections on  $P$ , then  $\mu_b^a = \hat{\omega}_b^a - \omega_b^a$  is a  $\mathfrak{g}$ -valued form on  $P$  with property that  $\mu(\sigma(a)) = 0$  for any  $a \in \mathfrak{g}$ . Then  $\mu_b^a = \mu_{bc}^a \theta^c$ .

$$(5) \quad d\theta^a = \hat{\omega}_b^a \wedge \theta^b + \hat{T}_{bc}^a \theta^b \wedge \theta^c = (\omega_b^a + \mu_{bc}^a \theta^c) \wedge \theta^b + \hat{T}_{bc}^a \theta^b \wedge \theta^c = \omega_b^a \wedge \theta^b + \left( \hat{T}_{bc}^a - \mu_{[bc]}^a \right) \theta^b \wedge \theta^c = \omega_b^a \wedge \theta^b + T_{bc}^a \theta^b \wedge \theta^c$$

Hence follows that

$$(6) \quad \hat{\omega}_b^a = \omega_b^a + \mu_{bc}^a \theta^c \Rightarrow \hat{T}_{bc}^a = T_{bc}^a + \mu_{[bc]}^a$$

Let us define the Spencer operator  $\delta$  from the space of tensors  $T_1^2(\mathbb{R}^n)$  of type  $(2, 1)$  to the space  $\Lambda^2(\mathbb{R}^n) \otimes \mathbb{R}^n$  as follows:

$$(7) \quad \delta : t_{bc}^a \in T_1^2(\mathbb{R}^n) \mapsto t_{[bc]}^a = \frac{1}{2}(t_{bc}^a - t_{cb}^a).$$

Note that  $\mathfrak{g} \otimes (\mathbb{R}^n)^* \subset \mathfrak{gl}(n) \otimes \mathbb{R}^* \cong T_1^2(\mathbb{R}^n)$  and we will denote the restriction of  $\delta$  to  $\mathfrak{g} \otimes (\mathbb{R}^n)^*$  by the same letter  $\delta$ . Thus, (6) can be rewritten as follows:

$$(8) \quad \hat{\omega}_b^a = \omega_b^a + \mu_{bc}^a \theta^c \Rightarrow \hat{T}_{bc}^a = T_{bc}^a + \delta(\mu_{bc}^a).$$

From (8) we conclude that if  $\delta : \mathfrak{g} \otimes (\mathbb{R}^n)^* \rightarrow \Lambda^2(\mathbb{R}^n) \otimes \mathbb{R}^n$  is a monomorphism, then, pseudoconnections  $\omega_b^a, \hat{\omega}_b^a$  with the same torsion  $T_{bc}^a$  coincide.

Now denote

$$(9) \quad \mathcal{T} = \frac{\Lambda^2(\mathbb{R}^n) \otimes \mathbb{R}^n}{\delta(\mathfrak{g} \otimes (\mathbb{R}^n)^*)}.$$

From (8) it follows that one can correctly define the *structure function*:

$$(10) \quad \mathcal{C} : P \rightarrow \mathcal{T}, \quad \xi \mapsto [T_{bc}^a(\xi)].$$

*G*-equivariance of structure function. The group  $G$  acts on  $\Lambda^2(\mathbb{R}^n) \otimes \mathbb{R}^n$  from the right as follows:

$$(11) \quad (\bar{\rho}(g)T)_{bc}^a = \tilde{g}_r^a T_{pq}^r g_b^p g_c^q$$

and one can easily prove that the subspace  $\delta(\mathfrak{g} \otimes (\mathbb{R}^n)^*)$  is invariant under this action. Then we have the following  $G$ -action on  $\mathcal{T}$ :

$$(12) \quad \forall g \in G, \quad \rho(g) : \mathcal{T} \rightarrow \mathcal{T}, \quad [T_{bc}^a] \mapsto [\tilde{g}_r^a T_{pq}^r g_b^p g_c^q]$$

By cumbersome calculations, from the structure equations (4) one can obtain that

$$(13) \quad \mathcal{C}(\xi g) = \mathcal{C}(g^{-1}\xi) = \rho(g)\mathcal{C}(\xi), \quad \forall \xi \in P, g \in G.$$

**Remark 1.** If  $\omega$  is a connection, one can prove that  $T_{bc}^a(\xi g) = \tilde{g}_r^a T_{pq}^r(\xi) g_b^p g_c^q$ , however it is wrong if  $\omega$  is a pseudoconnection. In this case, we have only that  $T_{bc}^a(\xi g) = \tilde{g}_r^a T_{pq}^r(\xi) g_b^p g_c^q + \nu_{bc}^a$ , where  $\nu_{bc}^a \in \delta(\mathfrak{g} \otimes (\mathbb{R}^n)^*)$ .

1.1.2. *Cartan reduction.* Let  $P \rightarrow M$  be a  $G$ -structure. Let  $\mathcal{T} = \sqcup \mathcal{T}_\alpha$  be the decomposition of  $\mathcal{T}$  into orbits of the  $G$ -action (12). Assume that the structure function  $c$  takes values in one orbit  $\mathcal{T}_0$ , only.

Fix  $\tau_0 \in \mathcal{T}_0$ . Then

$$(14) \quad P_1 = \{\xi \mid \mathcal{C}(\xi) = \tau_0\}$$

is the total space of a principal  $G_1$ -subbundle of  $P$ , where

$$(15) \quad G_1 = \{g \in G \mid \rho(g)\tau_0 = \tau_0\}.$$

They say that the  $G_1$ -structure  $P_1 \rightarrow M$  is obtained by the *Cartan reduction* from the  $G$ -structure  $P \rightarrow M$ .

1.2.  **$G$ -structure associated to a sub-Riemannian surface.** Let  $\mathcal{S} = (M, \Delta, \langle \cdot, \cdot \rangle)$  be a sub-Riemannian surface. We say that a coframe  $\eta = (\eta^1, \eta^2, \eta^3)$  of  $T_p M$ ,  $p \in M$ , is adapted to  $\mathcal{S}$  if

- (1)  $\eta$  is positively oriented, and  $(\eta^1|_{\Delta_p}, \eta^2|_{\Delta_p})$  is a positively oriented coframe of  $\Delta_p$ ;
- (2)  $\eta^3 \in \text{Ann}(\Delta)_p$ , or, equivalently,  $\eta^3(W) = 0$  for any  $W \in \Delta_p$ ;
- (3)  $\langle W, W \rangle = [\eta^1(W)]^2 + [\eta^2(W)]^2$  for any  $W \in \Delta_p$ .

To a given sub-Riemannian surface  $\mathcal{S} = (M, \Delta, \langle \cdot, \cdot \rangle)$  we associate the principal subbundle  $B_0 \subset B$  consisting of adapted frames. It is clear that the structure group of  $B_0$  is

$$G_0 = \left\{ \begin{pmatrix} A & B \\ 0 & c \end{pmatrix} \mid A \in SO(2), B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathbf{R}^2, c \in \mathbf{R} \setminus \{0\} \right\}.$$

One can easily prove

**Proposition 1.** *A sub-Riemannian surface  $\mathcal{S} = (M, \Delta, \langle \cdot, \cdot \rangle)$  is equivalent to a  $G_0$ -structure on  $M$ .*

1.2.1. *Contact sub-Riemannian surfaces.*

Contact distributions. Let  $\omega$  be a 1-form on a  $(2n+1)$ -dimensional manifold  $M$ . The form  $\omega$  is said to be *contact* if

$$(16) \quad \underbrace{\omega \wedge d\omega \wedge \dots \wedge d\omega}_n \neq 0.$$

A 1-form is contact if and only if  $\omega$  is nonvanishing (hence the Pfaff equation  $\omega = 0$  determines a  $2n$ -dimensional distribution  $\Delta$ ), and  $d\omega|_{\Delta}$  is nondegenerate.

Let  $\Delta$  be a  $2n$ -dimensional distribution on a  $(2n+1)$ -dimensional manifold  $M$ . Denote by  $\text{Ann}(\Delta)$  the vector subbundle in  $T^*M$  of rank 1 such that the fiber of  $\text{Ann}(\Delta)$  at  $p \in M$  is

$$(17) \quad \text{Ann}(\Delta)_p = \{\omega \in T_p^*M \mid \omega(W) = 0 \ \forall W \in \Delta_p\}.$$

The distribution  $\Delta$  is said to be *contact* if for each  $p \in M$  there exists a contact section  $\omega$  of  $\text{Ann}(\Delta)$  in a neighborhood of  $p$ . Note that this definition does not

depend on the choice of  $\omega$ : if  $\Delta$  is contact, then any nonvanishing section of  $\text{Ann}(\Delta)$  is contact.

We say that a sub-Riemannian surface  $\mathcal{S} = (M, \Delta, \langle \cdot, \cdot \rangle)$  is *contact* if the distribution  $\Delta$  is contact.

**Theorem 1.** *Any contact sub-Riemannian surface  $\mathcal{S} = (M, \Delta, \langle \cdot, \cdot \rangle)$  uniquely determines an  $SO(2)$ -structure  $B_2 \rightarrow M$  and a connection on  $B_2$  with the connection form*

$$(18) \quad \omega = \begin{pmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The 1-form  $\alpha$  in (18) together with the tautological forms  $\theta^a$  give a coframe field on  $B_2$ . The structure equations (4) are written as follows:

$$(19) \quad \begin{pmatrix} d\theta^1 \\ d\theta^2 \\ d\theta^3 \end{pmatrix} = \begin{pmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \wedge \begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{pmatrix} + \begin{pmatrix} a_1 & a_2 & 0 \\ a_2 & -a_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \theta^2 \wedge \theta^3 \\ \theta^3 \wedge \theta^1 \\ \theta^1 \wedge \theta^2 \end{pmatrix}$$

This theorem was proved by K. Hughen in [12] (also a sketch of the proof is given in [14], Ch. 7, 7.10). We present here a detailed proof of this theorem which is based on the Cartan reduction procedure as it was exposed in 1.1.1 and 1.1.2.

**1.2.2. Proof of Theorem 1. Step 1.** We start with the  $G_0$ -structure  $B_0 \rightarrow M$  associated with  $\mathcal{S}$  (see Proposition 1). We have

$$(20) \quad G_0 = \left\{ \begin{pmatrix} \cos \varphi & -\sin \varphi & b_1 \\ \sin \varphi & \cos \varphi & b_2 \\ 0 & 0 & c \end{pmatrix} \mid c \neq 0 \right\}$$

Then the Lie algebra of  $G_0$  is

$$(21) \quad \mathfrak{g}_0 = \left\{ \begin{pmatrix} 0 & \alpha & \beta_1 \\ -\alpha & 0 & \beta_2 \\ 0 & 0 & \gamma \end{pmatrix} \mid \alpha, \beta_1, \beta_2, \gamma \in \mathbb{R} \right\}$$

Now we will calculate

$$(22) \quad \mathcal{T}_0 = \frac{\Lambda^2(\mathbb{R}^3) \otimes \mathbb{R}^3}{\delta(\mathfrak{g}_0 \otimes (\mathbb{R}^3)^*)},$$

(see (9)). Let us denote the standard basis of  $\mathbb{R}^3$  by  $e_1, e_2, e_3$ , and the dual basis by  $e^1, e^2, e^3$ . Then the basis of  $\mathfrak{g}_0$  is

$$(23) \quad \mathcal{E}_1 = e_1 \otimes e^2 - e_2 \otimes e^1; \mathcal{E}_2 = e_1 \otimes e^3; \mathcal{E}_3 = e_2 \otimes e^3; \mathcal{E}_4 = e_3 \otimes e^3;$$

and  $\mathcal{E}_i \otimes e^a$ ,  $i = \overline{1, 4}$ ,  $a = \overline{1, 3}$ , is the basis of  $\mathfrak{g}_0 \otimes (\mathbb{R}^3)^*$ . Then  $\delta : \mathfrak{g}_0 \otimes (\mathbb{R}^n)^* \rightarrow \Lambda^2(\mathbb{R}^n) \otimes \mathbb{R}^n$  acts on the basis elements as follows

$$(24) \quad \begin{cases} \mathcal{E}_1 \otimes e^1 \mapsto -e_1 \otimes e^2 \wedge e^1, & \mathcal{E}_1 \otimes e^2 \mapsto -e_2 \otimes e^1 \wedge e^2, \\ \mathcal{E}_1 \otimes e^3 \mapsto e_1 \otimes e^2 \wedge e^3 - e_2 \otimes e^1 \wedge e^3, \\ \mathcal{E}_2 \otimes e^1 \mapsto e_1 \otimes e^3 \wedge e^1, & \mathcal{E}_2 \otimes e^2 \mapsto e_1 \otimes e^3 \wedge e^2, & \mathcal{E}_2 \otimes e^3 \mapsto 0, \\ \mathcal{E}_3 \otimes e^1 \mapsto e_2 \otimes e^3 \wedge e^1, & \mathcal{E}_3 \otimes e^2 \mapsto e_2 \otimes e^3 \wedge e^2, & \mathcal{E}_3 \otimes e^3 \mapsto 0, \\ \mathcal{E}_4 \otimes e^1 \mapsto e_3 \otimes e^3 \wedge e^1, & \mathcal{E}_4 \otimes e^2 \mapsto e_3 \otimes e^3 \wedge e^2, & \mathcal{E}_4 \otimes e^3 \mapsto 0, \end{cases}$$

From (24) we get that  $\mathcal{T}_0$  is spanned by  $[e_3 \otimes e^1 \wedge e^2]$  and so is one-dimensional.

Now let us find the action of  $G_0$  on  $\mathcal{T}_0$ . From (12) we get that, for any  $g \in G_0$ ,

$$(25) \quad \rho(g)[e_3 \otimes e^1 \wedge e^2] = [\tilde{g}_3^a g_b^1 g_c^2 e_a \otimes e^b \wedge e^c] = [\tilde{g}_3^3 (g_1^1 g_2^2 - g_1^2 g_2^1) e_3 \otimes e^1 \wedge e^2] = c^{-1} [e_3 \otimes e^1 \wedge e^2].$$

Hence the action of  $G_0$  on  $\mathcal{T}_0$  has two orbits:  $\mathcal{O}_0 = \{0 \in \mathcal{T}_0\}$  and  $\mathcal{O}_1 = \{t \in \mathcal{T}_0 \mid t \neq 0\}$ .

Let us prove that, if  $\mathcal{S}$  is contact, the structure function  $\mathcal{C}$  takes values in  $\mathcal{O}_1$ . The structure equations (4) can be written as follows :

$$(26) \quad \begin{pmatrix} d\theta^1 \\ d\theta^2 \\ d\theta^3 \end{pmatrix} = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ 0 & 0 & \delta \end{pmatrix} \wedge \begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{pmatrix} + \begin{pmatrix} T_{23}^1 & T_{31}^1 & T_{12}^1 \\ T_{23}^2 & T_{31}^2 & T_{12}^2 \\ T_{23}^3 & T_{31}^3 & T_{12}^3 \end{pmatrix} \begin{pmatrix} \theta^2 \wedge \theta^3 \\ \theta^3 \wedge \theta^1 \\ \theta^1 \wedge \theta^2 \end{pmatrix}$$

Now take a section  $s : U \rightarrow B_0$ , that is a coframe field  $\eta^a$  adapted to  $\mathcal{S}$  on  $U$ . Then from the definition of the tautological forms we have that  $ds^* \theta^a = \eta^a$ , hence  $ds^*(d\theta^3 \wedge \theta^3) = d\eta^3 \wedge \eta^3 \neq 0$  because  $\eta^3$  is a contact form (see (16)). At the same time, from (26), we get that  $d\theta^3 \wedge \theta^3 = T_{12}^3 \theta^1 \wedge \theta^2 \wedge \theta^3$ , hence follows that  $T_{12}^3 \neq 0$ . Thus  $\mathcal{C}(s(p)) = [T_{12}^3(s(p)) e_3 \otimes e^1 \wedge e^2] \neq 0$  for any  $p \in U$ . As each  $\xi \in \pi^{-1}(U)$  can be written as  $\xi = s(p)g$ ,  $p = \pi(\xi)$ , and the structure function  $\mathcal{C}$  satisfies (13), we have that  $\mathcal{C}(\xi) \neq 0$  for any  $\xi \in B_0$ , hence  $\mathcal{C}$  takes values in  $\mathcal{O}_1$ .

Thus, we can make the Cartan reduction and pass to the  $G_1$ -structure  $B_1 \rightarrow M$ , where

$$(27) \quad B_1 = \{\xi \mid \mathcal{C}(\xi) = [e_3 \otimes e^1 \wedge e^2]\}$$

is the total space of a principal  $G_1$ -subbundle of  $B_0$ , and

$$(28) \quad G_1 = \{g \in G \mid \rho(g)[e_3 \otimes e^1 \wedge e^2] = [e_3 \otimes e^1 \wedge e^2]\} = \left\{ \begin{pmatrix} \cos \varphi & -\sin \varphi & b_1 \\ \sin \varphi & \cos \varphi & b_2 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

**Step 2.** The Lie algebra of  $G_1$  is

$$(29) \quad \mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & \alpha & \beta_1 \\ -\alpha & 0 & \beta_2 \\ 0 & 0 & 0 \end{pmatrix} \mid \alpha, \beta_1, \beta_2 \in \mathbb{R} \right\}$$

and, by the construction of  $B_1$ , the structure equations have the form:

$$(30) \quad \begin{pmatrix} d\theta^1 \\ d\theta^2 \\ d\theta^3 \end{pmatrix} = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ 0 & 0 & 0 \end{pmatrix} \wedge \begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{pmatrix} + \begin{pmatrix} T_{23}^1 & T_{31}^1 & T_{12}^1 \\ T_{23}^2 & T_{31}^2 & T_{12}^2 \\ T_{23}^3 & T_{31}^3 & 1 \end{pmatrix} \begin{pmatrix} \theta^2 \wedge \theta^3 \\ \theta^3 \wedge \theta^1 \\ \theta^1 \wedge \theta^2 \end{pmatrix}$$

With notation of the previous step we have the basis of  $\mathfrak{g}_1$  is

$$(31) \quad \mathcal{E}_1 = e_1 \otimes e^2 - e_2 \otimes e^1; \mathcal{E}_2 = e_1 \otimes e^3; \mathcal{E}_3 = e_2 \otimes e^3;$$

and  $\mathcal{E}_i \otimes e^a$ ,  $i = \overline{1, 3}$ ,  $a = \overline{1, 3}$ , is the basis of  $\mathfrak{g}_1 \otimes (\mathbb{R}^3)^*$ . Then  $\delta : \mathfrak{g}_1 \otimes (\mathbb{R}^n)^* \rightarrow \Lambda^2(\mathbb{R}^n) \otimes \mathbb{R}^n$  acts on the basis elements as follows

$$(32) \quad \begin{cases} \mathcal{E}_1 \otimes e^1 \mapsto -e_1 \otimes e^2 \wedge e^1, & \mathcal{E}_1 \otimes e^2 \mapsto -e_2 \otimes e^1 \wedge e^2, \\ \mathcal{E}_1 \otimes e^3 \mapsto e_1 \otimes e^2 \wedge e^3 - e_2 \otimes e^1 \wedge e^3, \\ \mathcal{E}_2 \otimes e^1 \mapsto e_1 \otimes e^3 \wedge e^1, & \mathcal{E}_2 \otimes e^2 \mapsto e_1 \otimes e^3 \wedge e^2, & \mathcal{E}_2 \otimes e^3 \mapsto 0, \\ \mathcal{E}_3 \otimes e^1 \mapsto e_2 \otimes e^3 \wedge e^1, & \mathcal{E}_3 \otimes e^2 \mapsto e_2 \otimes e^3 \wedge e^2, & \mathcal{E}_3 \otimes e^3 \mapsto 0, \end{cases}$$

From (32) we get that

$$(33) \quad \mathcal{T}_1 = \frac{\Lambda^2(\mathbb{R}^3) \otimes \mathbb{R}^3}{\delta(\mathfrak{g}_1 \otimes (\mathbb{R}^3)^*)},$$

is spanned by  $[e_3 \otimes e^2 \wedge e^3]$ ,  $[e_3 \otimes e^3 \wedge e^1]$ ,  $[e_3 \otimes e^1 \wedge e^2]$  and so is three-dimensional.

At the same time, by construction of  $B_1$ , the structure function  $\mathcal{C}$  takes values in the affine subspace

$$(34) \quad \mathcal{T}'_1 = \{u[e_3 \otimes e^2 \wedge e^3] + v[e_3 \otimes e^3 \wedge e^1] + [e_3 \otimes e^1 \wedge e^2]\} \subset \mathcal{T}_1$$

Let us find the action of  $G_1$  on  $\mathcal{T}_1$ . Using (12), we find that

$$\begin{aligned} \rho(g)[e_3 \otimes e^2 \wedge e^3] &= \cos \varphi [e_3 \otimes e^2 \wedge e^3] - \sin \varphi [e_3 \otimes e^3 \wedge e^1] \\ \rho(g)[e_3 \otimes e^3 \wedge e^1] &= \sin \varphi [e_3 \otimes e^2 \wedge e^3] + \cos \varphi [e_3 \otimes e^3 \wedge e^1] \\ \rho(g)[e_3 \otimes e^1 \wedge e^2] &= (-b_1 \cos \varphi - b_2 \sin \varphi)[e_3 \otimes e^2 \wedge e^3] + \\ &\quad (b_1 \sin \varphi - b_2 \cos \varphi)[e_3 \otimes e^3 \wedge e^1] + [e_3 \otimes e^1 \wedge e^2] \end{aligned}$$

From this follows that  $\rho(g)$  maps  $\mathcal{T}'_1$  into itself, and moreover,

$$(35) \quad \begin{aligned} \rho(g)(u[e_3 \otimes e^2 \wedge e^3] + v[e_3 \otimes e^3 \wedge e^1] + [e_3 \otimes e^1 \wedge e^2]) &= \\ \{(u - b_1) \cos \varphi + (v - b_2) \sin \varphi\}[e_3 \otimes e^2 \wedge e^3] + \\ \{-(u - b_1) \sin \varphi + (v - b_2) \cos \varphi\}[e_3 \otimes e^3 \wedge e^1] + [e_3 \otimes e^1 \wedge e^2] \end{aligned}$$

Thus, we can make the Cartan reduction and pass to the  $G_2$ -structure  $B_2 \rightarrow M$ . We take

$$(36) \quad \tau_1 = 0 \cdot [e_3 \otimes e^2 \wedge e^3] + 0 \cdot [e_3 \otimes e^3 \wedge e^1] + [e_3 \otimes e^1 \wedge e^2] \in \mathcal{T}_1$$

and set

$$(37) \quad B_2 = \{\xi \mid \mathcal{C}(\xi) = \tau_1\}$$

is the total space of a principal  $G_2$ -subbundle of  $B_1$ , and

$$(38) \quad G_2 = \{g \in G_1 \mid \rho(g)\tau_1 = \tau_1\} = \left\{ \begin{pmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

**Step 3.** The Lie algebra of  $G_2$  is

$$(39) \quad \mathfrak{g}_2 = \left\{ \begin{pmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid \alpha \in \mathbb{R} \right\}$$

and the structure function  $\mathcal{C}(\xi) = \tau_1$ , for any  $\xi \in B_2$ , hence the structure equations are written as follows:

$$(40) \quad \begin{pmatrix} d\theta^1 \\ d\theta^2 \\ d\theta^3 \end{pmatrix} = \begin{pmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \wedge \begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{pmatrix} + \begin{pmatrix} T_{23}^1 & T_{31}^1 & T_{12}^1 \\ T_{23}^2 & T_{31}^2 & T_{12}^2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \theta^2 \wedge \theta^3 \\ \theta^3 \wedge \theta^1 \\ \theta^1 \wedge \theta^2 \end{pmatrix}$$

The basis of  $\mathfrak{g}_2$  is

$$(41) \quad \mathcal{E}_1 = e_1 \otimes e^2 - e_2 \otimes e^1;$$

and  $\mathcal{E}_1 \otimes e^a$ ,  $a = \overline{1, 3}$ , is the basis of  $\mathfrak{g}_2 \otimes (\mathbb{R}^3)^*$ . Then  $\delta : \mathfrak{g}_2 \otimes (\mathbb{R}^3)^* \rightarrow \Lambda^2(\mathbb{R}^3) \otimes \mathbb{R}^3$  acts on the basis elements as follows

$$(42) \quad \begin{cases} \mathcal{E}_1 \otimes e^1 \mapsto e_1 \otimes e^1 \wedge e^2, & \mathcal{E}_1 \otimes e^2 \mapsto -e_2 \otimes e^1 \wedge e^2, \\ \mathcal{E}_1 \otimes e^3 \mapsto e_1 \otimes e^2 \wedge e^3 + e_2 \otimes e^3 \wedge e^1, \end{cases}$$

Hence follows immediately that  $\delta$  is a monomorphism.

From (42) we get that

$$(43) \quad \mathcal{T}_2 = \frac{\Lambda^2(\mathbb{R}^3) \otimes \mathbb{R}^3}{\delta(\mathfrak{g}_2 \otimes (\mathbb{R}^3)^*)},$$

is spanned by

$$\begin{aligned} & [e_1 \otimes e^2 \wedge e^3] - [e_2 \otimes e^3 \wedge e^1], [e_1 \otimes e^3 \wedge e^1], [e_2 \otimes e^2 \wedge e^3], \\ & [e_3 \otimes e^2 \wedge e^3], [e_3 \otimes e^3 \wedge e^1], [e_3 \otimes e^1 \wedge e^2] \end{aligned}$$

and so is six-dimensional.

However, by construction of  $B_2$ , the structure function  $\mathcal{C}$  takes values in the affine subspace

$$(44) \quad \mathcal{T}_2' = \{u([e_1 \otimes e^2 \wedge e^3] - [e_2 \otimes e^3 \wedge e^1]) + v[e_1 \otimes e^3 \wedge e^1] + w[e_2 \otimes e^2 \wedge e^3] + [e_3 \otimes e^1 \wedge e^2]\} \subset \mathcal{T}_2$$

Hence follows that the structure equations have the following form

$$(45) \quad \begin{pmatrix} d\theta^1 \\ d\theta^2 \\ d\theta^3 \end{pmatrix} = \begin{pmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \wedge \begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{pmatrix} + \begin{pmatrix} u & v & 0 \\ w & -u & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \theta^2 \wedge \theta^3 \\ \theta^3 \wedge \theta^1 \\ \theta^1 \wedge \theta^2 \end{pmatrix}$$

Let us now prove that the form  $\alpha$  which satisfies (45) is unique, In  $\Lambda^2(\mathbb{R}^3) \otimes \mathbb{R}^3$  consider subspace  $N$  spanned by

$$\begin{aligned} & e_1 \otimes e^2 \wedge e^3 - e_2 \otimes e^3 \wedge e^1, e_1 \otimes e^3 \wedge e^1, e_2 \otimes e^2 \wedge e^3, \\ & e_3 \otimes e^2 \wedge e^3, e_3 \otimes e^3 \wedge e^1, e_3 \otimes e^1 \wedge e^2. \end{aligned}$$

It is clear that we have the direct sum

$$(46) \quad \Lambda^2(\mathbb{R}^3) \otimes \mathbb{R}^3 = N \oplus \delta(\mathfrak{g}_2 \otimes (\mathbb{R}^3)^*)$$

and  $\{T_{bc}^a\}$  from (45) takes values in  $N$ , If we have another  $\hat{\alpha}$  and the corresponding torsion  $\{\hat{T}_{bc}^a\}$  which satisfy (45), then  $\{\hat{T}_{bc}^a\}$  also take values in  $N$ , so the same is true for  $\{\hat{T}_{bc}^a - T_{bc}^a\}$ . However, from (6) it follows that  $\hat{T}_{bc}^a - T_{bc}^a = \delta(\mu_{bc}^a)$ . As we have the direct sum decomposition (46), we obtain that  $\hat{T}_{bc}^a - T_{bc}^a = 0$  and  $\delta(\mu_{bc}^a) = 0$ . But  $\delta$  is a monomorphism (see (42)), hence  $\mu_{bc}^a = 0$ , and so  $\hat{\omega}_b^a = \omega_b^a$  (see (6)). Thus  $\hat{\alpha} = \alpha$ .

To finish the proof of the theorem it is sufficient to prove that in (45)  $v = w$ . From (45) we get  $d\theta^3 = \theta^1 \wedge \theta^2$ , then, again using (45), we obtain

$$(47) \quad 0 = d\theta^1 \wedge \theta^2 - \theta^1 \wedge d\theta^2 = v\theta^3 \wedge \theta^1 \wedge \theta^2 - w\theta^1 \wedge \theta^2 \wedge \theta^3 = (v - w)\theta^1 \wedge \theta^2 \wedge \theta^3.$$

Now we set  $u = a_1$ ,  $v = w = a_2$  and from (45) get the structure equations (19).

Thus we have proved that for any contact sub-Riemannian surface  $\mathcal{S} = (M, \Delta, \langle \cdot, \cdot \rangle)$  there exists an  $SO(2)$ -structure on  $M$  and a unique pseudoconnection form  $\alpha$  such that the structure equations (19) hold true. The uniqueness of the  $SO(2)$ -structure  $B_2$  will be proved later, in Corollary 1 of Proposition 2.

**1.2.3. The functions  $a_1$ ,  $a_2$  and 1-form  $\alpha$  in terms of structure functions of a local frame.** Let  $\eta = \{\eta^a\}$  be a coframe field in a neighborhood  $U$  of  $p \in M$  which is a section of  $B_2 \rightarrow M$ . Let  $d\eta^a = C_{bc}^a \eta^b \wedge \eta^c$  be the corresponding structure equations. Then, for  $\bar{\eta}^a = d\pi^* \eta^a$ , we have  $d\bar{\eta}^a = \bar{C}_{bc}^a \bar{\eta}^b \wedge \bar{\eta}^c$ , where  $\bar{C}_{bc}^a = \pi^* C_{bc}^a = C_{bc}^a \circ \pi : \pi^{-1}(U) \rightarrow \mathbb{R}$ .

Let  $\psi : \pi^{-1}(U) \rightarrow U \times SO(2)$  be a local trivialization of  $\pi : B_2 \rightarrow M$  determined by  $\eta$ , then

$$(48) \quad \psi^{-1}(p, g(\varphi)) = g(\varphi)^{-1} \eta_p,$$

where

$$(49) \quad g(\varphi) = \begin{pmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \eta_p = \begin{pmatrix} \eta_p^1 \\ \eta_p^2 \\ \eta_p^3 \end{pmatrix}.$$

**Proposition 2.** a) If a coframe field  $\eta = \{\eta^a\}$  is a local section of  $B_2 \rightarrow M$ , then  $d\eta^3 = \eta^1 \wedge \eta^2$ .

b) The form  $\alpha$  is expressed in terms of  $\bar{C}_{bc}^a$  as follows:

$$(50) \quad \alpha = d\varphi + \bar{C}_{12}^1 \bar{\eta}^1 + \bar{C}_{12}^2 \bar{\eta}^2 - \frac{1}{2} (\bar{C}_{23}^1 + \bar{C}_{31}^2) \bar{\eta}^3$$

c) The functions  $a_1$  and  $a_2$  are expressed in terms of  $\bar{C}_{bc}^a$  as follows:

$$(51) \quad a_1 = \cos 2\varphi \left( \frac{\bar{C}_{23}^1 - \bar{C}_{31}^2}{2} \right) + \sin 2\varphi (\bar{C}_{31}^1)$$

$$(52) \quad a_2 = -\sin 2\varphi \left( \frac{\bar{C}_{23}^1 - \bar{C}_{31}^2}{2} \right) + \cos 2\varphi (\bar{C}_{31}^1)$$

*Proof.* If a coframe field  $\eta = \{\eta^a\}$  is a local section of  $B_2 \rightarrow M$ , then the equations (3) are written as follows:

$$(53) \quad \begin{cases} \theta^1 = \cos \varphi \bar{\eta}^1 + \sin \varphi \bar{\eta}^2 \\ \theta^2 = -\sin \varphi \bar{\eta}^1 + \cos \varphi \bar{\eta}^2 \\ \theta^3 = \bar{\eta}^3 \end{cases} \iff \begin{cases} \bar{\eta}^1 = \cos \varphi \theta^1 - \sin \varphi \theta^2 \\ \bar{\eta}^2 = \sin \varphi \theta^1 + \cos \varphi \theta^2 \\ \bar{\eta}^3 = \theta^3 \end{cases}$$

Then

$$(54) \quad \begin{cases} \theta^2 \wedge \theta^3 = \sin \varphi \bar{\eta}^3 \wedge \bar{\eta}^1 + \cos \varphi \bar{\eta}^2 \wedge \bar{\eta}^3 \\ \theta^3 \wedge \theta^1 = \cos \varphi \bar{\eta}^3 \wedge \bar{\eta}^1 - \sin \varphi \bar{\eta}^2 \wedge \bar{\eta}^3 \\ \theta^1 \wedge \theta^2 = \bar{\eta}^1 \wedge \bar{\eta}^2 \end{cases}$$

$$(55) \quad \begin{cases} d\bar{\eta}^1 = -\sin \varphi d\varphi \wedge \theta^1 + \cos \varphi d\theta^1 - \cos \varphi d\varphi \wedge \theta^2 - \sin \varphi d\theta^2 \\ d\bar{\eta}^2 = \cos \varphi d\varphi \wedge \theta^1 + \sin \varphi d\theta^1 - \sin \varphi d\varphi \wedge \theta^2 + \cos \varphi d\theta^2 \\ d\bar{\eta}^3 = d\theta^3 \end{cases}$$

To (55) we substitute  $d\theta^a$  from (19) and then  $\theta^a$  from (53) and use (54), finally we arrive at

$$(56) \quad \begin{cases} d\bar{\eta}^1 = (\alpha - d\varphi) \wedge \bar{\eta}^2 + (a_1 \cos 2\varphi - a_2 \sin 2\varphi) \bar{\eta}^2 \wedge \bar{\eta}^3 + \\ \quad (a_1 \sin 2\varphi + a_2 \cos 2\varphi) \bar{\eta}^3 \wedge \bar{\eta}^1 \\ d\bar{\eta}^2 = (d\varphi - \alpha) \wedge \bar{\eta}^1 + (a_1 \sin 2\varphi + a_2 \cos 2\varphi) \bar{\eta}^2 \wedge \bar{\eta}^3 + \\ \quad (-a_1 \cos 2\varphi + a_2 \sin 2\varphi) \bar{\eta}^3 \wedge \bar{\eta}^1 \\ d\bar{\eta}^3 = \bar{\eta}^1 \wedge \bar{\eta}^2 \end{cases}$$

Let us set  $\alpha - d\varphi = p_1 \bar{\eta}^1 + p_2 \bar{\eta}^2 + p_3 \bar{\eta}^3 + p_4 d\varphi$  and substitute to (56). We get

$$(57) \quad \begin{cases} d\bar{\eta}^1 = (a_1 \cos 2\varphi - a_2 \sin 2\varphi - p_3) \bar{\eta}^2 \wedge \bar{\eta}^3 + (a_1 \sin 2\varphi + a_2 \cos 2\varphi) \bar{\eta}^3 \wedge \bar{\eta}^1 + \\ \quad p_1 \bar{\eta}^1 \wedge \bar{\eta}^2 + p_4 d\varphi \wedge \bar{\eta}^2 \\ d\bar{\eta}^2 = (a_1 \sin 2\varphi + a_2 \cos 2\varphi) \bar{\eta}^2 \wedge \bar{\eta}^3 + (-a_1 \cos 2\varphi + a_2 \sin 2\varphi - p_3) \bar{\eta}^3 \wedge \bar{\eta}^1 + \\ \quad p_2 \bar{\eta}^1 \wedge \bar{\eta}^2 + p_4 \bar{\eta}^1 \wedge d\varphi \\ d\bar{\eta}^3 = \bar{\eta}^1 \wedge \bar{\eta}^2 \end{cases}$$

At the same time,  $d\bar{\eta}^a = \bar{C}_{bc}^a \bar{\eta}^b \wedge \bar{\eta}^c$ , hence we get that  $p_4 = 0$  and  $\bar{C}_{bc}^a$  are expressed as follows:

$$(58) \quad \bar{C}_{23}^1 = a_1 \cos 2\varphi - a_2 \sin 2\varphi - p_3$$

$$(59) \quad \bar{C}_{31}^1 = a_1 \sin 2\varphi + a_2 \cos 2\varphi$$

$$(60) \quad \bar{C}_{12}^1 = p_1$$

$$(61) \quad \bar{C}_{23}^2 = a_1 \sin 2\varphi + a_2 \cos 2\varphi$$

$$(62) \quad \bar{C}_{31}^2 = -a_1 \cos 2\varphi + a_2 \sin 2\varphi - p_3$$

$$(63) \quad \bar{C}_{12}^2 = p_2$$

$$(64) \quad \bar{C}_{23}^3 = 0$$

$$(65) \quad \bar{C}_{31}^3 = 0$$

$$(66) \quad \bar{C}_{12}^3 = 1$$

From (64)–(66) we get claim a).

The equations (60), (63), and the sum of equations (58), (62) give us  $p_1$ ,  $p_2$ , and  $p_3 = -\frac{1}{2}(\bar{C}_{23}^1 + \bar{C}_{31}^2)$ , thus we prove claim b).

The equations (58), (59) with  $p_3$  substituted give us claim c),

Finally note that (59), (61) imply that  $\bar{C}_{31}^1 = \bar{C}_{23}^2$ , this also can be proved in the following way. We have  $d\bar{\eta}^3 = \bar{\eta}^1 \wedge \bar{\eta}^2$ . From this it follows that  $0 = d\bar{\eta}^1 \wedge \bar{\eta}^2 - \bar{\eta}^1 \wedge d\bar{\eta}^2$ , hence we obtain  $\bar{C}_{31}^1 = \bar{C}_{23}^2$ .  $\square$

**Corollary 1.** *The  $SO(2)$ -structure  $B_2 \rightarrow M$ , where  $B_2$  is a  $SO(2)$ -principal subbundle of  $B_0$  such that the tautological forms  $\theta^a$  on  $B_2$  satisfy structure equations (19) is unique.*

*Proof.* Let  $B_2$  and  $\hat{B}_2$  be  $SO(2)$ -principal subbundles of  $B_0$  such that the tautological forms satisfy structure equations (19). Take local sections  $\eta^a$  and  $\hat{\eta}^a$  of  $B_2$  and  $\hat{B}_2$ , respectively.

By Proposition 2 a), we have

$$(67) \quad d\eta^3 = \eta^1 \wedge \eta^2 \text{ and } d\hat{\eta}^3 = \hat{\eta}^1 \wedge \hat{\eta}^2$$

Let  $\Omega$  be the area form on  $\Delta$  determined by the metric  $\langle \cdot, \cdot \rangle$ . Since  $\eta^a$  and  $\hat{\eta}^a$  are sections of  $B_0$ , we have

$$\eta^1 \wedge \eta^2|_{\Delta} = \Omega = \hat{\eta}^1 \wedge \hat{\eta}^2|_{\Delta}$$

By the same reason, we have  $\hat{\eta}^3 = e^f \eta^3$ , hence  $d\hat{\eta}^3 = e^f df \wedge \eta^3 + e^f d\eta^3$ . We restrict it to  $\Delta$  and from  $d\hat{\eta}^3|_{\Delta} = d\eta^3|_{\Delta}$  get that  $e^f = 1$ . Hence  $\hat{\eta}^3 = \eta^3$ . Now

$$(68) \quad \hat{\eta}^1 = \cos \varphi \eta^1 - \sin \varphi \eta^2 + a \eta^3$$

$$(69) \quad \hat{\eta}^2 = \sin \varphi \eta^1 + \cos \varphi \eta^2 + b \eta^3$$

But

$$\hat{\eta}^1 \wedge \hat{\eta}^2 = d\hat{\eta}^3 = d\eta^3 = \eta^1 \wedge \eta^2,$$

so one can easily prove that  $a = b = 0$ .  $\square$

**Corollary 2.** *The function*

$$(70) \quad \bar{\mathcal{M}} = (a_1)^2 + (a_2)^2 = \left( \frac{\bar{C}_{23}^1 - \bar{C}_{31}^2}{2} \right)^2 + (\bar{C}_{31}^1)^2$$

*is a pullback of a function  $\mathcal{M} : M \rightarrow \mathbb{R}$ , i. e.  $\bar{\mathcal{M}} = \mathcal{M} \circ \pi$ , where*

$$(71) \quad \mathcal{M} = \left( \frac{C_{23}^1 - C_{31}^2}{2} \right)^2 + (C_{31}^1)^2.$$

From Corollary 1 it follows that the  $SO(2)$ -structure  $B_2$  is uniquely determined by the sub-Riemannian surface  $\mathcal{S}$ . As the coframe field  $\{\theta^1, \theta^2, \theta^3, \alpha\}$  on  $B_2$  is uniquely determined, we see that the functions  $a_1, a_2 : B_2 \rightarrow \mathbb{R}$  as well as the form  $\alpha$  are uniquely determined by  $\mathcal{S}$ . Thus we get

**Corollary 3.** *The function  $\mathcal{M}$  is an invariant of the sub-Riemannian surface  $\mathcal{S}$ .*

1.2.4. *Curvature of a contact sub-Riemannian surface.* Let us write down (19) as follows:

$$(72) \quad d\theta^1 = \alpha \wedge \theta^2 + a_1\theta^2 \wedge \theta^3 + a_2\theta^3 \wedge \theta^1$$

$$(73) \quad d\theta^2 = -\alpha \wedge \theta^1 + a_2\theta^2 \wedge \theta^3 - a_1\theta^3 \wedge \theta^1$$

$$(74) \quad d\theta^3 = \theta^1 \wedge \theta^2$$

Take the exterior differential of (72), then we get

$$(75) \quad 0 = d\alpha \wedge \theta^2 - \alpha \wedge d\theta^2 + da_1 \wedge \theta^2 \wedge \theta^3 + a_1 d\theta^2 \wedge \theta^3 - a_1 \theta^2 \wedge d\theta^3 + da_2 \wedge \theta^3 \wedge \theta^1 + a_2 d\theta^3 \wedge \theta^1 - a_2 \theta^3 \wedge d\theta^1$$

To (75) we substitute (72)–(74) and get

$$(76) \quad d\alpha \wedge \theta^2 + 2a_1\alpha \wedge \theta^3 \wedge \theta^1 - 2a_2\alpha \wedge \theta^2 \wedge \theta^3 + da_1 \wedge \theta^2 \wedge \theta^3 + da_2 \wedge \theta^3 \wedge \theta^1 = 0.$$

In the same manner from (73) we get

$$(77) \quad -d\alpha \wedge \theta^1 + 2a_1\alpha \wedge \theta^2 \wedge \theta^3 + 2a_2\alpha \wedge \theta^3 \wedge \theta^1 + da_2 \wedge \theta^2 \wedge \theta^3 - da_1 \wedge \theta^3 \wedge \theta^1 = 0.$$

Now we consider the expansions:

$$(78) \quad \begin{cases} da_1 = a\alpha + A_1\theta^1 + A_2\theta^2 + A_3\theta^3 \\ da_2 = b\alpha + B_1\theta^1 + B_2\theta^2 + B_3\theta^3 \\ d\alpha = P_1\alpha \wedge \theta^1 + P_2\alpha \wedge \theta^2 + P_3\alpha \wedge \theta^3 + X_{23}\theta^2 \wedge \theta^3 + X_{31}\theta^3 \wedge \theta^1 + X_{12}\theta^1 \wedge \theta^2. \end{cases}$$

We will express  $P_a$  and  $X_{ab}$  in terms of  $A_a$  and  $B_b$ . To do it we substitute (78) to (76) and get

$$(79) \quad \begin{aligned} &P_1\alpha \wedge \theta^1 \wedge \theta^2 + P_3\alpha \wedge \theta^3 \wedge \theta^2 + X_{31}\theta^3 \wedge \theta^1 \wedge \theta^2 + 2a_1\alpha \wedge \theta^3 \wedge \theta^1 - \\ &2a_2\alpha \wedge \theta^2 \wedge \theta^3 + a\alpha \wedge \theta^2 \wedge \theta^3 + A_1\theta^1 \wedge \theta^2 \wedge \theta^3 + \\ &b\alpha \wedge \theta^3 \wedge \theta^1 + B_2\theta^2 \wedge \theta^3 \wedge \theta^1 = 0. \end{aligned}$$

From this we get

$$(80) \quad P_1 = 0, \quad P_3 - a + 2a_2 = 0, \quad X_{31} + A_1 + B_2 = 0, \quad 2a_1 + b = 0.$$

In the same manner, substituting (78) to (77), we get

$$(81) \quad P_2 = 0, \quad -P_3 - a + 2a_2 = 0, \quad -X_{23} + B_1 - A_2 = 0, \quad 2a_1 + b = 0.$$

From (80) and (81) we get

$$(82) \quad \begin{aligned} P_1 = P_2 = P_3 = 0, \quad a = 2a_2, \quad b = -2a_1, \\ X_{31} = -A_1 - B_2, \quad X_{23} = B_1 - A_2. \end{aligned}$$

Thus only  $X_{12}$  is undetermined, and we denote it by  $\bar{\mathcal{K}}$ . In this way we obtain

$$(83) \quad \begin{cases} da_1 = 2a_2\alpha + A_1\theta^1 + A_2\theta^2 + A_3\theta^3 \\ da_2 = -2a_1\alpha + B_1\theta^1 + B_2\theta^2 + B_3\theta^3 \\ d\alpha = \bar{\mathcal{K}}\theta^1 \wedge \theta^2 + (B_1 - A_2)\theta^2 \wedge \theta^3 + (-A_1 - B_2)\theta^3 \wedge \theta^1 \end{cases}$$

Now let us express  $\bar{\mathcal{K}}$  in terms of  $\bar{C}_{ab}^c$ . To do it, we use (53) and (54). Then, from (50) we get

$$(84) \quad \begin{aligned} d\alpha = d\bar{C}_{12}^1 \wedge \bar{\eta}^1 + \bar{C}_{12}^1 \wedge d\bar{\eta}^1 + d\bar{C}_{12}^2 \wedge \bar{\eta}^2 + \bar{C}_{12}^2 \wedge d\bar{\eta}^2 - \\ \frac{1}{2}d(\bar{C}_{23}^1 + \bar{C}_{31}^2) \wedge \bar{\eta}^3 - \frac{1}{2}(\bar{C}_{23}^1 + \bar{C}_{31}^2) \wedge d\bar{\eta}^3. \end{aligned}$$

Let us take the frame field  $\{\bar{E}_1, \bar{E}_2, \bar{E}_3, \bar{E}_4\}$  dual to  $\{\bar{\eta}^1, \bar{\eta}^2, \bar{\eta}^3, \alpha\}$ . Note that  $\bar{C}_{ab}^c$  depend only on the base coordinates, so  $E_4 \bar{C}_{ab}^c = 0$ . Therefore,

$$(85) \quad \begin{aligned} d\bar{C}_{23}^1 &= \bar{E}_1 \bar{C}_{23}^1 \bar{\eta}^1 + \bar{E}_2 \bar{C}_{23}^1 \bar{\eta}^2 + \bar{E}_3 \bar{C}_{23}^1 \bar{\eta}^3; \\ d\bar{C}_{23}^2 &= \bar{E}_1 \bar{C}_{23}^2 \bar{\eta}^1 + \bar{E}_2 \bar{C}_{23}^2 \bar{\eta}^2 + \bar{E}_3 \bar{C}_{23}^2 \bar{\eta}^3. \end{aligned}$$

If we substitute (85) to (84) we obtain expansion

$$(86) \quad d\alpha = (\bar{E}_1 \bar{C}_{12}^2 - \bar{E}_2 \bar{C}_{12}^1 + (\bar{C}_{12}^1)^2 + (\bar{C}_{12}^2)^2 - \frac{1}{2}(\bar{C}_{23}^1 + \bar{C}_{31}^2))\bar{\eta}^1 \wedge \bar{\eta}^2 + (\dots)\bar{\eta}^3 \wedge \bar{\eta}^1 + (\dots)\bar{\eta}^2 \wedge \bar{\eta}^3$$

where  $\dots$  stands for the coefficient we are not interested in now. Then, use (54), and from (86) we get

$$(87) \quad d\alpha = (\bar{E}_1 \bar{C}_{12}^2 - \bar{E}_2 \bar{C}_{12}^1 + (\bar{C}_{12}^1)^2 + (\bar{C}_{12}^2)^2 - \frac{1}{2}(\bar{C}_{23}^1 + \bar{C}_{31}^2))\theta^1 \wedge \theta^2 + (\dots)\theta^3 \wedge \theta^1 + (\dots)\theta^2 \wedge \theta^3$$

Compare (83) and (86), then we finally find

$$(88) \quad \bar{\mathcal{K}} = \bar{E}_1 \bar{C}_{12}^2 - \bar{E}_2 \bar{C}_{12}^1 + (\bar{C}_{12}^1)^2 + (\bar{C}_{12}^2)^2 - \frac{1}{2}(\bar{C}_{23}^1 + \bar{C}_{31}^2).$$

It is clear that  $\bar{E}_a$  are horizontal lifts of vector fields  $E_a$  which constitute a local frame field on  $U$ , and  $\bar{E}_a \bar{C}_{cd}^b = (E_a C_{cd}^b) \circ \pi$ . As  $\bar{C}_{bc}^a = C_{bc}^a \circ \pi$ , we have  $\bar{\mathcal{K}} = \mathcal{K} \circ \pi$ , where

$$(89) \quad \mathcal{K} = E_1 C_{12}^2 - E_2 C_{12}^1 + (C_{12}^1)^2 + (C_{12}^2)^2 - \frac{1}{2}(C_{23}^1 + C_{31}^2).$$

The function  $\mathcal{K}$  is called the *curvature* of  $\mathcal{S}$ , and it is clear that  $\mathcal{K}$  is an invariant of  $\mathcal{S}$ .

We result our investigations of invariants of a contact sub-Riemannian surface in the following

**Theorem 2.** *Let  $\mathcal{S} = (M, \Delta, \langle \cdot, \cdot \rangle)$  be a contact sub-Riemannian surface. Then, for any  $p \in M$ , in a neighborhood  $U$  of  $p$  a coframe field  $\eta = \{\eta^a\}$  exists such that*

$$(90) \quad \begin{pmatrix} d\eta^1 \\ d\eta^2 \\ d\eta^3 \end{pmatrix} = \begin{pmatrix} C_{23}^1 & C_{31}^1 & C_{12}^1 \\ C_{23}^2 & C_{31}^2 & C_{12}^2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta^2 \wedge \eta^3 \\ \eta^3 \wedge \eta^1 \\ \eta^1 \wedge \eta^2 \end{pmatrix}$$

*The functions*

$$(91) \quad \mathcal{M} = \left( \frac{C_{23}^1 - C_{31}^2}{2} \right)^2 + (C_{31}^1)^2$$

$$(92) \quad \mathcal{K} = E_1 C_{12}^2 - E_2 C_{12}^1 + (C_{12}^1)^2 + (C_{12}^2)^2 - \frac{1}{2}(C_{23}^1 + C_{31}^2)$$

*do not depend on the choice of coframe field  $\eta$  with structure equations (90) and are correctly defined on  $M$ .*

## 2. SYMMETRIES OF CONTACT SUB-RIEMANNIAN SURFACES

A *symmetry* of a sub-Riemannian surface  $\mathcal{S} = (M, \Delta, \langle \cdot, \cdot \rangle)$  is a local diffeomorphism  $F : M \rightarrow M$  such that, for any  $p \in M$ ,

$$(93) \quad F(\Delta_p) = \Delta_{F(p)},$$

$$(94) \quad \langle dF(W), dF(W) \rangle_{F(p)} = \langle W, W \rangle_p, \forall W \in \Delta_p.$$

A vector field  $V \in \mathfrak{X}(M)$  is called an *infinitesimal symmetry* if its flow consists of symmetries.

**Theorem 3.** *Let  $\mathcal{S} = (M, \Delta, \langle \cdot, \cdot \rangle)$  be a contact sub-Riemannian surface. Let  $\eta = \{\eta^a\}$  be a coframe field in a neighborhood  $U$  of  $p \in M$  such that (90) holds true ( $\eta$  exists by Theorem 2), and  $\{E_a\}$  be the dual frame field.*

*a) For any infinitesimal symmetry  $V$  of  $\mathcal{S}$ , a unique function  $f : U \rightarrow \mathbb{R}$  exists such that*

$$(95) \quad V = -E_2(f)E_1 + E_1(f)E_2 + fE_3 \text{ and } E_3f = 0.$$

*b) Let  $\mathcal{M}$  and  $\mathcal{K}$  be the invariants of  $\mathcal{S}$  ( see Theorem 2). Then, if  $V$  is transversal to  $\Delta$  and  $E_1\mathcal{K}E_2\mathcal{M} - E_2\mathcal{K}E_1\mathcal{M} \neq 0$ , the function  $\ln f$  satisfies the following system of partial differential equations:*

$$(96) \quad \begin{cases} E_1(\ln f) = \frac{E_3\mathcal{K}E_1\mathcal{M} - E_1\mathcal{K}E_3\mathcal{M}}{E_1\mathcal{K}E_2\mathcal{M} - E_2\mathcal{K}E_1\mathcal{M}} \\ E_2(\ln f) = \frac{E_3\mathcal{K}E_2\mathcal{M} - E_2\mathcal{K}E_3\mathcal{M}}{E_1\mathcal{K}E_2\mathcal{M} - E_2\mathcal{K}E_1\mathcal{M}} \\ E_3(\ln f) = 0. \end{cases}$$

*Proof.* Let  $V$  be an infinitesimal symmetry of  $\mathcal{S}$ , and  $\phi_t$  be the flow of  $V$ . As  $\{E_1(p), E_2(p)\}$  is an orthonormal frame of  $\Delta(p)$ , we have, by definition of infinitesimal symmetry (93), (94), that

$$(97) \quad \begin{cases} d\phi_t E_1(p) = \cos \varphi(t) E_1(\phi_t(p)) + \sin \varphi(t) E_2(\phi_t(p)) \\ d\phi_t E_2(p) = -\sin \varphi(t) E_1(\phi_t(p)) + \cos \varphi(t) E_2(\phi_t(p)) \end{cases}$$

Hence

$$(98) \quad \begin{cases} d\phi_t E_1(\phi_{-t}(p)) = \cos \varphi(t) E_1(p) + \sin \varphi(t) E_2(p) \\ d\phi_t E_2(\phi_{-t}(p)) = -\sin \varphi(t) E_1(p) + \cos \varphi(t) E_2(p) \end{cases}$$

From this follows that

$$(99) \quad \begin{cases} [E_1, V] = \lambda E_2 \\ [E_2, V] = -\lambda E_1 \end{cases}$$

because

$$(100) \quad \mathcal{L}_V E_1 = [V, E_1] = \frac{d}{dt} \Big|_{t=0} d\phi_t E_1(\phi_{-t}(p)) = \varphi'(0) E_2 = -\lambda E_2$$

and similar for  $[V, E_2] = \lambda E_1$ .

On the other hand we know that the structure equations for the dual frame  $E = (E_1, E_2, E_3)$  are

$$(101) \quad \begin{cases} [E_1, E_2] = c_{12}^1 E_1 + c_{12}^2 E_2 + c_{12}^3 E_3 \\ [E_3, E_1] = c_{31}^1 E_1 + c_{31}^2 E_2 + c_{31}^3 E_3 \\ [E_2, E_3] = c_{23}^1 E_1 + c_{23}^2 E_2 + c_{23}^3 E_3 \end{cases}$$

But  $c_{jk}^i = -C_{jk}^i$  from (90). Therefore

$$(102) \quad \begin{cases} [E_1, E_2] = -(C_{12}^1 E_1 + C_{12}^2 E_2 + C_{12}^3 E_3) \\ [E_3, E_1] = -(C_{31}^1 E_1 + C_{31}^2 E_2 + C_{31}^3 E_3) \\ [E_2, E_3] = -(C_{23}^1 E_1 + C_{23}^2 E_2 + C_{23}^3 E_3) \end{cases}$$

Substituting  $V = V^1 E_1 + V^2 E_2 + V^3 E_3$  to the first equation in (99), we get

$$(103) \quad \begin{aligned} \lambda E_2 &= [E_1, V^1 E_1 + V^2 E_2 + V^3 E_3] = \\ &= E_1 V^1 E_1 + E_1 V^2 E_2 + V^2 [E_1, E_2] + E_1 V^3 E_3 + V^3 [E_1, E_3] = \\ &= (E_1 V^1 - V^2 C_{12}^1 + V^3 C_{31}^1) E_1 + (E_1 V^2 - V^2 C_{12}^2 + V^3 C_{31}^2) E_2 + (E_1 V^3 - V^2) E_3. \end{aligned}$$

In the same manner, substituting  $V = V^1 E_1 + V^2 E_2 + V^3 E_3$  to the second equation in (99), we get

$$(104) \quad \begin{aligned} -\lambda E_1 &= [E_2, V^1 E_1 + V^2 E_2 + V^3 E_3] = \\ &= E_2 V^1 E_1 + V^1 [E_2, E_1] + E_2 V^2 E_2 + E_2 V^3 E_3 + V^3 [E_2, E_3] = \\ &= (E_2 V^1 + V^1 C_{12}^1 - V^3 C_{23}^1) E_1 + (E_2 V^2 + V^1 C_{12}^2 - V^3 C_{23}^2) E_2 + (E_2 V^3 + V^1) E_3. \end{aligned}$$

From (103) and (104) we obtain the following equation system:

$$(105) \quad E_1 V^1 - V^2 C_{12}^1 + V^3 C_{31}^1 = 0,$$

$$(106) \quad E_1 V^2 - V^2 C_{12}^2 + V^3 C_{31}^2 = \lambda,$$

$$(107) \quad E_1 V^3 - V^2 = 0,$$

$$(108) \quad E_2 V^1 + V^1 C_{12}^1 - V^3 C_{23}^1 = -\lambda,$$

$$(109) \quad E_2 V^2 + V^1 C_{12}^2 - V^3 C_{23}^2 = 0,$$

$$(110) \quad E_2 V^3 + V^1 = 0.$$

Let us set  $f = \eta^3(V) = V^3$ , then (107) and (110) give

$$(111) \quad V = -E_2(f)E_1 + E_1(f)E_2 + fE_3.$$

Now substitute  $V^1 = -E_2 f$ ,  $V^2 = E_1 f$ , and  $V^3 = f$  to (105) and (109):

$$-E_1 E_2 f - C_{12}^1 E_1 f + C_{31}^1 f = 0,$$

$$E_2 E_1 f - C_{12}^2 E_2 f - C_{23}^2 f = 0,$$

Summing these equalities, we arrive at

$$(112) \quad [E_2, E_1]f - C_{12}^1 E_1 f - C_{12}^2 E_2 f = 0$$

but, by the first equation in (102), this means that  $E_3 f = 0$ . Thus we have proved (95) and claim a).

Let us now prove b). As  $V$  is an infinitesimal symmetry of the sub-Riemannian surface  $\mathcal{S}$ , we have

$$(113) \quad \begin{cases} V\mathcal{K} = 0 \\ V\mathcal{M} = 0 \end{cases}$$

If we substitute (95) to (113), we get

$$(114) \quad \begin{cases} -E_1\mathcal{K}E_2f + E_2\mathcal{K}E_1f + E_3\mathcal{K}f = 0 \\ -E_1\mathcal{M}E_2f + E_2\mathcal{M}E_1f + E_3\mathcal{M}f = 0 \end{cases}$$

Since  $V$  is transversal to  $\Delta$ , and hence  $f$  does not vanish in  $U$ , we can divide both equations by  $f$  and obtain the system of linear equations in  $E_1 \ln f$  and  $E_2 \ln f$ :

$$(115) \quad \begin{cases} -E_1\mathcal{K}E_2 \ln f + E_2\mathcal{K}E_1 \ln f + E_3\mathcal{K} = 0 \\ -E_1\mathcal{M}E_2 \ln f + E_2\mathcal{M}E_1 \ln f + E_3\mathcal{M} = 0 \end{cases}$$

If  $E_1\mathcal{K}E_2\mathcal{M} - E_2\mathcal{K}E_1\mathcal{M} \neq 0$ , this system has the unique solution

$$(116) \quad E_1(\ln f) = \frac{E_3\mathcal{K}E_1\mathcal{M} - E_1\mathcal{K}E_3\mathcal{M}}{E_1\mathcal{K}E_2\mathcal{M} - E_2\mathcal{K}E_1\mathcal{M}}$$

$$(117) \quad E_2(\ln f) = \frac{E_3\mathcal{K}E_2\mathcal{M} - E_2\mathcal{K}E_3\mathcal{M}}{E_1\mathcal{K}E_2\mathcal{M} - E_2\mathcal{K}E_1\mathcal{M}}$$

To (116) and (117) we add  $E_3 \ln f = 0$ , which follows from (95), and get the system (96). Thus we have proved b).  $\square$

**Remark 2.** *If an infinitesimal symmetry  $V$  of  $\mathcal{S}$  lies in  $\Delta$  at each point of an open set  $W$ , then  $f$  is zero, and, by (95),  $V$  is zero, too. So, nonvanishing  $V$  should be transversal to  $\Delta$  almost everywhere.*

**Remark 3.** *Theorem 3 can be used in order to prove that a sub-Riemannian surface  $\mathcal{S}$  does not admit nontrivial infinitesimal symmetries. To do it, it is sufficient to prove that the integrability conditions do not hold for the system (96). The integrability conditions have the form:*

$$(118) \quad \begin{cases} E_1(EQ2) - E_2(EQ1) = C_{12}^1EQ1 + C_{12}^2EQ2 \\ E_3EQ1 = C_{31}^1EQ1 + C_{31}^2EQ2 \\ -E_3EQ2 = C_{23}^1EQ1 + C_{23}^2EQ2, \end{cases}$$

where

$$\begin{aligned} EQ1 &= \frac{E_3\mathcal{K}E_1\mathcal{M} - E_1\mathcal{K}E_3\mathcal{M}}{E_1\mathcal{K}E_2\mathcal{M} - E_2\mathcal{K}E_1\mathcal{M}}, \\ EQ2 &= \frac{E_3\mathcal{K}E_2\mathcal{M} - E_2\mathcal{K}E_3\mathcal{M}}{E_1\mathcal{K}E_2\mathcal{M} - E_2\mathcal{K}E_1\mathcal{M}}. \end{aligned}$$

However, if the integrability conditions (118) do hold for (96), one can use this system in order to find  $f$  and then  $V$ . In fact, we can take a natural frame field  $\partial_a$  and write  $E_a = B_a^b \partial_b$ . Then the equation system (96) can be rewritten as  $\partial_a \ln f = g_a$ , and the solution can be found by the well-known formula:

$$(119) \quad \ln f(x^a) = \int_0^1 x^b g_b(tx^a) dt.$$

**Remark 4.** The condition  $\mathcal{D} = E_1\mathcal{K}E_2\mathcal{M} - E_2\mathcal{K}E_1\mathcal{M} \neq 0$ , in general, does not hold. If  $\mathcal{D} = 0$ , the system (115) may not have any solutions and this means that  $\mathcal{S}$  does not admit any infinitesimal symmetries; or it may have infinitely many solutions, then we simply get an additional relation for  $f$ , which can be used in order to find infinitesimal symmetries by another method.

## 2.1. Examples of infinitesimal symmetries.

2.1.1. *Heisenberg distribution.* Consider the Heisenberg distribution  $\Delta$  given with respect to the standard coordinates in  $\mathbb{R}^3$  by the 1-form

$$\eta^3 = dz + ydx - xdy.$$

For the metric on  $\Delta$  we take the metric induced from  $\mathbb{R}^3$ . By calculations, we get the following results:

(1) The  $SO(2)$ -structure  $B_2 \rightarrow \mathbb{R}^3$  is given by the coframe field

$$\begin{cases} \eta^1 = \frac{(2+3y^2)dx}{2\sqrt{1+y^2}} - \frac{3xydy}{2\sqrt{1+y^2}} + \frac{ydz}{2\sqrt{1+y^2}} \\ \eta^2 = -\frac{xydx}{2\sqrt{1+y^2}\sqrt{1+x^2+y^2}} + \frac{(2+3x^2+2y^2)dy}{2\sqrt{1+y^2}\sqrt{1+x^2+y^2}} - \frac{x dz}{2\sqrt{1+y^2}\sqrt{1+x^2+y^2}} \\ \eta^3 = -\frac{y}{2}\sqrt{1+x^2+y^2}dx + \frac{x}{2}\sqrt{1+x^2+y^2}dy - \frac{1}{2}\sqrt{1+x^2+y^2}dz \end{cases}$$

(2) The orthonormal dual frame is

$$\begin{cases} E_1 = \frac{1}{\sqrt{1+y^2}}\frac{\partial}{\partial x} - \frac{y}{\sqrt{1+y^2}}\frac{\partial}{\partial z} \\ E_2 = \frac{xy}{\sqrt{1+y^2}\sqrt{1+x^2+y^2}}\frac{\partial}{\partial x} + \frac{\sqrt{1+y^2}}{\sqrt{1+x^2+y^2}}\frac{\partial}{\partial y} + \frac{x}{\sqrt{1+y^2}\sqrt{1+x^2+y^2}}\frac{\partial}{\partial z} \\ E_3 = \frac{y}{(1+x^2+y^2)^{3/2}}\frac{\partial}{\partial x} - \frac{x}{(1+x^2+y^2)^{3/2}}\frac{\partial}{\partial y} - \frac{2+3x^2+3y^2}{(1+x^2+y^2)^{3/2}}\frac{\partial}{\partial z} \end{cases}$$

(3) The structure functions  $^1 C_{jk}^i$  are

$$\begin{aligned} C_{23}^1 &= -\frac{1-3y^2}{(1+y^2)(1+x^2+y^2)} & C_{23}^2 &= -\frac{3xy}{(1+y^2)(1+x^2+y^2)^{3/2}} & C_{23}^3 &= 0 \\ C_{31}^1 &= -\frac{3xy}{(1+y^2)(1+x^2+y^2)^{3/2}} & C_{31}^2 &= -\frac{1-2x^2+y^2}{(1+y^2)(1+x^2+y^2)} & C_{31}^3 &= 0 \\ C_{12}^1 &= -\frac{3y}{\sqrt{1+y^2}\sqrt{1+x^2+y^2}} & C_{12}^2 &= \frac{2x}{\sqrt{1+y^2}(1+x^2+y^2)} & C_{12}^3 &= 1 \end{aligned}$$

(4) The invariants

$$\begin{cases} \mathcal{M} = \frac{9}{4} \frac{(x^2+y^2)^2}{(1+x^2+y^2)^4} \\ \mathcal{K} = \frac{3(1+2x^2+4y^2+3x^2y^2+3y^4)}{(1+y^2)(1+x^2+y^2)^2} \end{cases}$$

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$^1 d\eta^i = C_{jk}^i \eta^j \wedge \eta^k$ ,  $[E_j, E_k] = c_{jk}^i E_i$ ,  $C_{jk}^i = -c_{jk}^i$

- (5) The family of functions  $f$  which define symmetries

$$f = A\sqrt{1+x^2+y^2}, \text{ where } A = \text{const.}$$

- (6) The connection form

$$\begin{aligned} \alpha &= d\varphi - \frac{1}{2(1+y^2)(1+x^2+y^2)^{3/2}} \{y(4+9x^2+16y^2+12x^2y^2+12y^4)dx \\ &- x(2+7x^2+14y^2+12x^2y^2+12y^4)dy + (-2+3x^2+4y^2+6x^2y^2+6y^4)dz\} \end{aligned}$$

**2.2. Cartan distribution.** Consider the Cartan distribution  $\Delta$  given with respect to the standard coordinates in  $\mathbf{R}^3$  by the 1-form

$$\eta^3 = dz + ydx.$$

- (1) The  $SO(2)$ -structure  $B_2 \rightarrow \mathbb{R}^3$  is given by the coframe field

$$\begin{cases} \eta^1 = \frac{1+2y^2}{\sqrt{1+y^2}}dx + \frac{y}{\sqrt{1+y^2}}dz \\ \eta^2 = dy \\ \eta^3 = -\sqrt{1+y^2}dx - \sqrt{1+y^2}dz \end{cases}$$

- (2) The orthonormal dual frame is

$$\begin{cases} E_1 = \frac{1}{\sqrt{1+y^2}}\frac{\partial}{\partial x} - \frac{y}{\sqrt{1+y^2}}\frac{\partial}{\partial z} \\ E_2 = \frac{\partial}{\partial y} \\ E_3 = \frac{y}{(1+y^2)^{3/2}} - \frac{1+2y^2}{(1+y^2)^{3/2}}\frac{\partial}{\partial z} \end{cases}$$

- (3) The structure functions are,

$$\begin{aligned} C_{23}^1 &= -\frac{1-y^2}{(1+y^2)^2} & C_{23}^2 &= 0 & C_{23}^3 &= 0 \\ C_{31}^1 &= 0 & C_{31}^2 &= 0 & C_{31}^3 &= 0 \\ C_{12}^1 &= -\frac{2y}{1+y^2} & C_{12}^2 &= 0 & C_{12}^3 &= 1 \end{aligned}$$

- (4) The invariants

$$\begin{cases} \mathcal{M} = \frac{1}{4} \frac{(1-2y^2)^2}{(1+y^2)^4} \\ \mathcal{K} = \frac{1+4y^2}{(1+y^2)^2} \end{cases}$$

- (5) The family of functions  $f$  which define symmetries

$$f = f(y, z + xy)$$

- (6) The connection form

$$\alpha = d\varphi - \frac{y(1+6y^2)}{(1+y^2)^{3/2}}dx + \frac{y(1-4y^2)}{(1+y^2)^{3/2}}dz$$

## 3. NONCONTACT SUB-RIEMANNIAN SURFACES OF STABLE TYPE

Let us consider a sub-Riemannian surface  $(\Delta, \langle \cdot, \cdot \rangle)$ . Now we do not assume that  $\Delta$  is contact, so we admit that the set

$$(120) \quad \Sigma = \{p \in \mathbf{R}^3 \mid (\omega \wedge d\omega)_p = 0\},$$

where  $\Delta$  is the kernel of the 1-form  $\omega$ , is, in general, non-empty. However, we assume that  $\Sigma$  is a 2-dimensional submanifold in  $\mathbf{R}^3$  and the distribution  $\Delta$  is transversal to  $\Sigma$ .

**Remark 5.** *The surface  $\Sigma$  does not depend on the choice of  $\omega$ .*

**Remark 6.** *Any stable germ of a Pfaffian equation on a 3-dimensional manifold is equivalent either to the germ of the 1-form  $\omega_0 = dz + xdy$ , or b)  $\omega_1 = dy + x^2dz$ , at the origin [23]. For a contact distribution  $\Delta$ , the germ of  $\Delta$  at each point is equivalent to the germ of the distribution determined by  $\omega_0$ . If a distribution  $\Delta$  satisfies our assumption, then, for any point  $p \in \mathbf{R}^3 \setminus \Sigma$ , the germ of  $\Delta$  at  $p$  is equivalent to the germ of the distribution determined by  $\omega_0$ , and, for  $p \in \Sigma$ , the germ is equivalent to the germ of the distribution determined by  $\omega_0$ .*

**3.1. Nonholonomy function of sub-Riemannian surface.** For the sub-Riemannian surface  $(\Delta, \langle \cdot, \cdot \rangle)$  let us take a non-vanishing section  $\omega$  of the bundle  $\text{Ann}(\Delta)$ . Then, in a neighborhood  $U$  of each point  $p \in M$  take an positively oriented orthonormal frame field  $\{E_1, E_2\}$  of  $\Delta$ . Then we define the function  $\lambda_U : U \rightarrow \mathbf{R}$ ,  $\lambda_U(q) = \omega([E_1, E_2](q))$ . One can easily check that  $\lambda_U$  does not depend on a choice of the frame field  $\{E_1, E_2\}$ , therefore if  $U \cap V \neq \emptyset$ ,  $\lambda_U|_{U \cap V} = \lambda_V|_{U \cap V}$ . Therefore, we have correctly defined function  $\lambda_\omega$  on  $M$  by setting  $\lambda_\omega|_U = \lambda_U$ . This function will be called the *nonholonomy function of sub-Riemannian surface*. Note that this function depends on the choice of form  $\omega$  and on the metric on  $\Delta$ .

**Proposition 3.** *The nonholonomy function has the following properties:*

- a)  $\lambda_{e^\varphi \omega} = e^\varphi \lambda_\omega$ ;
- b)  $\lambda(p) = 0$  if and only if  $p \in \Sigma$ ;
- c)  $d\lambda_\omega|_p \neq 0$  for any  $p \in \Sigma$ .
- d)  $d\omega|_\Delta = -\frac{1}{2}\lambda_\omega \Omega$ , where  $\Omega$  is the area 2-form on  $\Delta$  determined by the metric.

*Proof.* a) is evident from the definition of nonholonomy function.

b) In a neighborhood  $U$  of a point  $p$  take an positively oriented orthonormal frame field  $\{E_1, E_2\}$  of  $\Delta$  and a vector field  $E_3$  such that  $\omega(E_3) = 1$ . Then,  $\{E_1, E_2, E_3\}$  is a frame field on  $U$ . We have  $d\omega(E_1, E_2) = -\frac{1}{2}\omega([E_1, E_2]) = -\frac{1}{2}\lambda_\omega$ , and  $\omega(E_1) = \omega(E_2) = 0$ , from this follows

$$d\omega \wedge \omega(E_1, E_2, E_3) = -\frac{1}{6}\lambda_\omega.$$

This proves b).

c) By our assumptions, for any  $p \in \Sigma$ , with respect to a coordinate system,  $\omega = e^\varphi \omega_1$ , where  $\omega_1 = dz + x^2 dy$ . From a) it follows that  $\lambda_\omega = e^\varphi \lambda_{\omega_1}$ . Also,  $dx, dy, \omega_1$  is a coframe, hence  $dx \wedge dy \wedge \omega_1(E_1, E_2, E_3) \neq 0$ , then  $dx \wedge dy(E_1, E_2) \neq 0$ .

Therefore,  $\lambda_{\omega_1} = e^\psi x$ , for a function  $\psi$ , and, hence,  $\Sigma \cap U$  is given by the equation  $x = 0$ , and  $\lambda_\omega = e^{\varphi+\psi} x$ . From this immediately follows the required statement.

d) For a positively oriented orthonormal frame field  $\{E_1, E_2\}$  of  $\Delta$  we have  $\Omega(E_1, E_2) = 1$  and  $d\omega(E_1, E_2) = -\frac{1}{2}\omega([E_1, E_2]) = -\frac{1}{2}\lambda_\omega$ . This proves d).  $\square$

**3.2. Characteristic vector field.** Let us denote by  $\text{Ann}(\Delta)$  the vector subbundle in  $T^*M$  of rank 1 whose fiber at  $p \in M$  consists of 1-forms vanishing at  $\Delta_p$ .

**Proposition 4.** *For each point  $p \in M$  there exists a section  $\omega$  of  $\text{Ann}(\Delta)$  in a neighborhood  $U(p)$  which admits a vector field  $V$  on  $U(p)$  such that  $L_V\omega = 0$  and  $\omega(V) = 1$ . For a given  $\omega$  this vector field is unique.*

*Proof.* For  $p \in \Sigma$  we can take  $\omega = dz + x^2 dy$ , for the other  $p$  we can take  $\omega = dz + x dy$  with respect to an appropriate coordinate system (see Remark 6). In both cases, the vector field  $V = \frac{\partial}{\partial z}$  has the required properties.

Let us prove that, for a given  $\omega$ , the vector field  $V$  such that  $L_V\omega = 0$  and  $\omega(V) = 1$  is unique.

Let us take a frame field  $E_1, E_2, E_3$  on  $U(p)$  such that  $\Delta$  is spanned by  $E_1$  and  $E_2$ , and  $E_3 = V$ . Now let  $W$  be a vector field with the required properties, then  $W = W^1 E_1 + W^2 E_2 + W^3 E_3$ . Since  $\omega(V) = \omega(W) = 1$ , we have  $W^3 = 1$ . As  $E_3$  is an infinitesimal symmetry of  $\omega$ ,  $E_3$  is an infinitesimal symmetry of  $\Delta$ , too, therefore the vector fields  $[E_3, E_1], [E_3, E_2]$  are tangent to  $\Delta$ . From this follows that the vector fields  $W^1[E_1, E_2], W^2[E_1, E_2]$  are tangent to  $\Delta$ , but  $\omega([E_1, E_2]) \neq 0$  on  $U(p) \setminus \Omega$ , therefore  $W^1 = W^2 = 0$  on  $U(p) \setminus \Omega$ , and  $W^1$  and  $W^2$  vanish on  $U(p)$ . Thus  $W = E_3 = V$  and the uniqueness has been proved.  $\square$

If for  $p \in M$ , a nonvanishing form  $\omega \in \text{Ann}(\Delta)$  on a neighborhood  $U$  of  $p$  for which there exists a vector field  $V$  on such that  $L_V\omega = 0$  and  $\omega(V) = 1$  will be called a *special form at  $p$* , and  $V$  the *characteristic vector field of  $\omega$* . Note that, if  $\omega$  is special at each point of an open set  $U \subset M$ , then on  $U$  we have a unique vector field  $V$  such that  $L_V\omega = 0$  and  $\omega(V) = 1$ .

Let us consider the form  $\tilde{\omega} = e^\varphi \omega$ . In general,  $\tilde{\omega}$  is not special.

**Proposition 5.** *a) If  $p \in M \setminus \Omega$ , then any nonvanishing form  $\omega \in \text{Ann}(\Delta)$  is special.*

*b) If  $p \in \Sigma$  and  $\omega$  is special at  $p$ , then  $\tilde{\omega} = e^\varphi \omega$  is special at  $p$  if and only if on a neighborhood  $U$  of  $p$  we have  $d\varphi|_\Delta = \lambda_\omega \xi$ , where  $\xi$  is a nonvanishing 1-form on  $\Delta$  in  $U$ .*

*Proof.* First note that if  $V$  is a vector field corresponding to a special form  $\omega$ , then  $L_V\omega = d(\iota_V\omega) + \iota_V d\omega = \iota_V d\omega$  because  $\iota_V\omega = 1$ . Therefore,  $V$  is the characteristic vector field of  $\omega$  if and only if  $\omega(V) = 1$  and  $\iota_V d\omega = 0$ .

Let  $\omega$  be a special form on a neighborhood  $U$  of  $p$ , and  $V$  be the corresponding characteristic vector field. Let us take a form  $\tilde{\omega} = e^\varphi \omega$  and find conditions on  $\varphi$  for  $\tilde{\omega}$  to be a special form.

First,  $\tilde{\omega}(\tilde{V}) = 1$  if and only if  $\tilde{V} = e^{-\varphi}V + W$ , where  $W \in \mathfrak{X}(\Delta)$ , because  $\omega(V) = 1$ . Further, we have

$$(121) \quad d\tilde{\omega} = e^\varphi d\varphi \wedge \omega + e^\varphi d\omega = d\varphi \wedge \tilde{\omega} + e^\varphi d\omega.$$

Then

$$(122) \quad \iota_{\tilde{V}} d\tilde{\omega} = \iota_{\tilde{V}} d\varphi \omega - d\varphi \iota_{\tilde{V}} \omega + e^\varphi \iota_{\tilde{V}} d\omega = \tilde{V} \varphi \omega - d\varphi \omega(\tilde{V}) + \iota_V d\omega + e^\varphi \iota_W d\omega;$$

As  $V$  is the characteristic vector field of  $\omega$ , we have  $\iota_V d\omega = 0$ , hence follows

$$(123) \quad \iota_{\tilde{V}} d\tilde{\omega} = \tilde{V} \varphi \tilde{\omega} - d\varphi + e^\varphi \iota_W d\omega.$$

Assume that  $p$  does not lie in  $\Sigma$ , then, by Proposition 3, d), we have that  $d\omega|_\Delta$  is nondegenerate form, so one can find a unique  $W$  such that  $d\varphi(W') = e^\varphi \iota_W d\omega(W')$ . By (123), with this  $W$ ,  $\iota_{\tilde{V}} d\tilde{\omega} = 0$  on  $\Delta$ . Also, if we substitute  $V$  to the right hand side of (123), then we get  $\tilde{V} \varphi e^\varphi - V \varphi = 0$ , since  $\iota_W d\omega(V) = 2d\omega(W, V) = -\iota_V d\omega(W) = 0$ . Thus, we have found  $W$  such that the corresponding vector field  $\tilde{V} = e^{-\varphi}V + W$  is the characteristic vector field for  $\tilde{\omega}$  since  $\tilde{\omega}(\tilde{V}) = 1$  and  $\iota_{\tilde{V}} d\tilde{\omega} = 0$ . Thus, the form  $\tilde{\omega}$  is special, and we have proved a).

For  $p \in \Sigma$ ,  $d\omega|_\Delta = \lambda_\omega \Omega$ , where  $\Omega$  is the area form of the metric on  $\Delta$ . Thus, if  $\omega$  and  $\tilde{\omega}$  are special, by (123) we have that  $d\varphi|_\Delta = -\frac{1}{2}\lambda_\omega e^\varphi \Omega(W, \cdot)$ , therefore in this case we have that  $d\varphi|_\Delta = \lambda_\omega \xi$ , where  $\xi(W') = -\frac{1}{2}e^\varphi \Omega(W, W')$  is a nonzero 1-form on  $\Delta$ . Now, if  $d\varphi|_\Delta = \lambda_\omega \xi$ , then one can find  $W$  such that  $\xi(W') = -\frac{1}{2}e^\varphi \Omega(W, W')$  for any  $W' \in \mathfrak{X}(\Delta)$ . If we now set  $\tilde{V} = e^{-\varphi}V + W$ , then  $\iota_{\tilde{V}} d\tilde{\omega}(W') = 0$ , for any  $W' \in \mathfrak{X}(\Delta)$ . Also, as before, we have  $\iota_{\tilde{V}} d\tilde{\omega}(V) = 0$ , hence follows  $\iota_{\tilde{V}} d\tilde{\omega} = 0$ . Thus,  $\tilde{\omega}$  is special and we have proved b).  $\square$

**Remark 7.** *It is clear that any symmetry of the nonholonomic surface maps a special form to a special form.*

**3.3. Adapted frame.** Assume that the distribution  $\Delta$  is given by a special form  $\omega$ . In a neighborhood  $U$  of  $p \in M$  take a frame field constructed in the following way. Since  $\lambda_\omega$  vanishes at  $\Sigma$  and  $d\lambda_\omega \neq 0$  at  $\Sigma$ ,  $U$  is foliated by the level surfaces  $\Sigma_c = \lambda_\omega^{-1}(c) \cap U$ ,  $c \in (-\alpha, \beta)$ ,  $\alpha, \beta > 0$ , of  $\lambda_\omega$ . Moreover, since  $\Delta$  is transversal to  $\Sigma = \Sigma_0$ , then  $\Delta$  is transversal to  $\Sigma_c$ , too. We take  $E_1$  be the unit vector field of the line distribution  $\Delta \cap T\Sigma_c$  on  $U$ ,  $c \in (-\alpha, \beta)$ , (in fact, there are two such vector fields, they are opposite each other, we take one of them). The vector field  $E_2 \in \mathfrak{X}(\Delta)$  is such that  $E_1, E_2$  is positively oriented orthonormal frame field of  $\Delta$ . For  $E_3$  we take the characteristic vector field of  $\omega$ .

Now, in the structure equations  $[E_i, E_j] = c_{ij}^k E_k$ , we have  $c_{31}^3 = c_{23}^3 = 0$  because the flow of  $E_3$  maps  $\Delta$  to  $\Delta$ , and  $c_{12}^3 = \lambda_\omega$  by definition of  $\lambda_\omega$ .

Let  $\{\eta^1, \eta^2, \eta^3\}$  be the coframe dual to  $\{E_a\}$ . Note that  $\eta^3 = \omega$ . Then the structure equations are

$$(124) \quad d\eta^1 = C_{23}^1 \eta^2 \wedge \eta^3 + C_{31}^1 \eta^3 \wedge \eta^1 + C_{12}^1 \eta^1 \wedge \eta^2,$$

$$(125) \quad d\eta^2 = C_{23}^2 \eta^2 \wedge \eta^3 + C_{31}^2 \eta^3 \wedge \eta^1 + C_{12}^2 \eta^1 \wedge \eta^2,$$

$$(126) \quad d\eta^3 = -\lambda_\omega \eta^1 \wedge \eta^2,$$

From the coframe construction it follows that

$$(127) \quad d\lambda_\omega = \lambda_2\eta^2 + \lambda_3\eta^3,$$

as  $\lambda_1 = E_1\lambda_\omega = 0$ .

Applying the exterior differential to (126), we get that

$$0 = -d\lambda_\omega\eta^1 \wedge \eta^2 + \lambda_\omega d\eta^1 \wedge \eta^2 - \lambda_\omega\eta^1 \wedge d\eta^2 = (-\lambda_3 + \lambda_\omega(C_{31}^1 - C_{23}^2))\eta^1 \wedge \eta^2 \wedge \eta^3.$$

hence follows that

$$(128) \quad \lambda_3 = \lambda_\omega(C_{31}^1 - C_{23}^2).$$

Therefore  $\lambda_3 = 0$  on  $\Sigma$ , and, as  $d\lambda_\omega \neq 0$  on  $\Sigma$  we have that  $\lambda_2 = E_2\lambda \neq 0$  on  $\Sigma$ .

**3.4. Change of the coframe.** It is clear that the frame  $\{E_a\}$ , and so the coframe  $\{\eta^a\}$ , is uniquely determined by the special form  $\omega$ . Now let us find how the coframe and the structure equations transform under a change of the special form  $\omega$ .

Let us take two special forms  $\omega$  and  $\tilde{\omega} = e^\varphi\omega$ , where  $d\varphi|_\Delta = \lambda_\omega\xi$  (see Proposition 5, b)). Let  $\{\eta^a\}$  and  $\{\tilde{\eta}^a\}$  be the corresponding coframe fields. Then from construction we have:

$$(129) \quad \begin{aligned} \eta^1 &= \cos\alpha \tilde{\eta}^1 + \sin\alpha \tilde{\eta}^2 + a \tilde{\eta}^3 \\ \eta^2 &= -\sin\alpha \tilde{\eta}^1 + \cos\alpha \tilde{\eta}^2 + b \tilde{\eta}^3 \\ \eta^3 &= e^{-\varphi} \tilde{\eta}^3. \end{aligned}$$

From Proposition 5, b) it follows that  $d\varphi = \lambda_\omega\xi_1\eta^1 + \lambda_\omega\xi_2\eta^2 + \varphi_3\eta^3$ , Then

$$(130) \quad \begin{aligned} d\tilde{\eta}^3 &= d(e^\varphi\eta^3) = e^\varphi d\eta^3 + e^\varphi d\eta^3 = e^\varphi\lambda_\omega(\xi_1\eta^1 + \xi_2\eta^2) \wedge \eta^3 + e^\varphi(-\lambda_\omega\eta^1 \wedge \eta^2) \\ &= \lambda_\omega[(\xi_1\eta^1 + \xi_2\eta^2) \wedge \tilde{\eta}^3 - e^\varphi\eta^1 \wedge \eta^2]. \end{aligned}$$

From (130) it follows that

$$\xi_1\eta^1 + \xi_2\eta^2 = (\xi_1 \cos\alpha - \xi_2 \sin\alpha)\tilde{\eta}^1 + (\xi_1 \sin\alpha + \xi_2 \cos\alpha)\tilde{\eta}^2 + (a\xi_1 + b\xi_2)\tilde{\eta}^3.$$

then

$$(131) \quad (\xi_1\eta^1 + \xi_2\eta^2) \wedge \tilde{\eta}^3 = (\xi_1 \cos\alpha - \xi_2 \sin\alpha)\tilde{\eta}^1 \wedge \tilde{\eta}^3 + (\xi_1 \sin\alpha + \xi_2 \cos\alpha)\tilde{\eta}^2 \wedge \tilde{\eta}^3.$$

Also

$$(132) \quad \eta^1 \wedge \eta^2 = \tilde{\eta}^1 \wedge \tilde{\eta}^2 + (b \cos\alpha + a \sin\alpha)\tilde{\eta}^1 \wedge \tilde{\eta}^3 + (b \sin\alpha - a \cos\alpha)\tilde{\eta}^2 \wedge \tilde{\eta}^3.$$

The equation (126) written for the coframe  $\{\tilde{\eta}^a\}$  gives  $d\tilde{\eta}^3 = -\lambda_{\tilde{\omega}}\tilde{\eta}^1 \wedge \tilde{\eta}^2$ , and from Proposition 3 we have  $\lambda_{\tilde{\omega}} = e^\varphi\lambda_\omega$ . Hence follows  $d\tilde{\eta}^3 = -e^\varphi\lambda_\omega\tilde{\eta}^1 \wedge \tilde{\eta}^2$ . Thus (130), (131), and (132) together give the equation system

$$(133) \quad \begin{cases} \lambda_\omega[\xi_1 \cos\alpha - \xi_2 \sin\alpha - e^\varphi(b \cos\alpha + a \sin\alpha)] = 0 \\ \lambda_\omega[\xi_1 \sin\alpha - \xi_2 \cos\alpha - e^\varphi(b \sin\alpha - a \cos\alpha)] = 0 \end{cases}$$

As  $\lambda_\omega \neq 0$  almost everywhere in  $U$ , we have that the expressions in brackets in (133) vanish. Therefore, we arrive at the system

$$(134) \quad \begin{cases} (\xi_1 - e^\varphi b) \cos\alpha - (\xi_2 + e^\varphi a) \sin\alpha = 0 \\ (\xi_1 - e^\varphi b) \sin\alpha - (\xi_2 + e^\varphi a) \cos\alpha = 0 \end{cases}$$

Thus we have found

$$(135) \quad a = -e^{-\varphi}\xi_2, \quad b = e^{-\varphi}\xi_1.$$

Now it remains to find the function  $\alpha$  in (130). To this end we use (127). We have

$$(136) \quad d\lambda_{\tilde{\omega}} = d(e^{\varphi}\lambda_{\omega}) = e^{\varphi}\lambda_{\omega}d\varphi + e^{\varphi}d\lambda_{\omega} = e^{\varphi}\lambda_{\omega}(\lambda_{\omega}\xi_1\eta^1 + \lambda_{\omega}\xi_2\eta^2 + \varphi_3\eta^3) + e^{\varphi}(\lambda_2\eta^2 + \lambda_3\eta^3) = e^{\varphi}[\lambda_{\omega}^2\xi_1\eta^1 + (\lambda_{\omega}^2\xi_2 + \lambda_2)\eta^2 + (\lambda_{\omega}\varphi_3 + \lambda_3)\eta^3].$$

To (136) we substitute (130) and get that

$$(137) \quad d\lambda_{\tilde{\omega}} = e^{\varphi}[\lambda_{\omega}^2\xi_1 \cos \alpha - (\lambda_{\omega}^2\xi_2 + \lambda_2) \sin \alpha]\tilde{\eta}^1 + e^{\varphi}[\lambda_{\omega}^2\xi_1 \sin \alpha + (\lambda_{\omega}^2\xi_2 + \lambda_2) \cos \alpha]\tilde{\eta}^2 + (\lambda_{\omega}\varphi_3 + \lambda_2\xi_1 + \lambda_3)\tilde{\eta}^3.$$

From this follows that

$$(138) \quad d\lambda_{\tilde{\omega}} = \tilde{\lambda}_1\tilde{\eta}^1 + \tilde{\lambda}_2\tilde{\eta}^2 + \tilde{\lambda}_3\tilde{\eta}^3,$$

where

$$(139) \quad \tilde{\lambda}_1 = e^{\varphi}[\lambda_{\omega}^2\xi_1 \cos \alpha - (\lambda_{\omega}^2\xi_2 + \lambda_2) \sin \alpha],$$

$$(140) \quad \tilde{\lambda}_2 = e^{\varphi}[\lambda_{\omega}^2\xi_1 \sin \alpha + (\lambda_{\omega}^2\xi_2 + \lambda_2) \cos \alpha]$$

$$(141) \quad \tilde{\lambda}_3 = \lambda_{\omega}\varphi_3 + \lambda_2\xi_1 + \lambda_3$$

From (128) it follows that  $\lambda_3 = \lambda_{\omega}f$ , in the same way,  $\tilde{\lambda}_3 = \lambda_{\tilde{\omega}}\tilde{f}$ , where  $f, \tilde{f}$  are functions. Then  $\tilde{\lambda}_3 = e^{\varphi}\lambda_{\omega}\tilde{f}$ . As  $\lambda_2 \neq 0$  at  $\Sigma$  (see reasoning below (128)), from (141) we have that

$$(142) \quad \xi_1 = \lambda_{\omega}\mu.$$

Also, by (127),  $\tilde{\lambda}_1 = 0$ , hence (139) and (142) give us the expression for  $\alpha$ :

$$(143) \quad \tan \alpha = \frac{\lambda_{\omega}^3\mu}{\lambda_{\omega}^2\xi_2 + \lambda_2}.$$

Thus we have proved

**Proposition 6.** *Let  $\{\eta^a\}$  be the adapted frame determined by a special form  $\omega$ , and  $\{\tilde{\eta}^a\}$  be the adapted frame determined by a special form  $\tilde{\omega} = e^{\varphi}\omega$ . Then*

$$(144) \quad \eta^1 = \cos \alpha \tilde{\eta}^1 + \sin \alpha \tilde{\eta}^2 - e^{-\varphi}\xi_2 \tilde{\eta}^3$$

$$(145) \quad \eta^2 = -\sin \alpha \tilde{\eta}^1 + \cos \alpha \tilde{\eta}^2 + e^{-\varphi}\xi_1 \tilde{\eta}^3$$

$$(146) \quad \eta^3 = e^{-\varphi}\tilde{\eta}^3,$$

Here functions  $\xi_1, \xi_2$ , and  $\alpha$  are determined by  $\varphi$  in the following way:

$$(147) \quad E_1\varphi = \varphi_1 = \lambda_{\omega}\xi_1 = \lambda_{\omega}^2\mu$$

$$(148) \quad E_2\varphi = \varphi_2 = \lambda_{\omega}\xi_2$$

$$(149) \quad \tan \alpha = \frac{\lambda_{\omega}^3\mu}{\lambda_{\omega}^2\xi_2 + \lambda_2},$$

where  $\{E_a\}$  is the adapted frame dual to  $\{\eta^a\}$ ,  $\mu$  is a function, and  $\lambda_2 = E_2\lambda_{\omega}$ .

**3.5. Invariants of sub-Riemannian surface along the singular surface.** Let  $\{\eta^a\}$  be the adapted frame determined by a special form  $\omega$ , and  $\{\tilde{\eta}^a\}$  be the adapted frame determined by a special form  $\tilde{\omega} = e^\varphi \omega$ . Then we have the structure equations:  $d\eta^a = C_{bc}^a \eta^b \wedge \eta^c$  and  $d\tilde{\eta}^a = \tilde{C}_{bc}^a \tilde{\eta}^b \wedge \tilde{\eta}^c$ . Let us denote the restrictions of the structure functions  $C_{bc}^a$  and  $\tilde{C}_{bc}^a$  to the surface  $\Sigma$  by  $Q_{bc}^a$  and  $\tilde{Q}_{bc}^a$ , respectively. Let us find relation between  $Q_{bc}^a$  and  $\tilde{Q}_{bc}^a$ ,

The surface  $\Sigma$  is given by the equation  $\lambda_\omega = 0$ . Using Proposition 6, we get that at the points of  $\Sigma$  the equalities (144)–(146) are written as follows:

$$(150) \quad \eta^1 = \tilde{\eta}^1 - e^{-\varphi} \xi_2 \tilde{\eta}^3$$

$$(151) \quad \eta^2 = \tilde{\eta}^2$$

$$(152) \quad \eta^3 = e^{-\varphi} \tilde{\eta}^3,$$

Now, let us take the exterior derivative of (144)

$$(153) \quad d\eta^1 = -\sin \alpha d\alpha \wedge \tilde{\eta}^1 + \cos \alpha d\tilde{\eta}^1 + \cos \alpha d\alpha \wedge \tilde{\eta}^2 + \sin \alpha d\tilde{\eta}^2 + e^{-\varphi} \xi_2 d\varphi \wedge \tilde{\eta}^3 - e^{-\varphi} d\xi_2 \wedge \tilde{\eta}^3 - e^{-\varphi} \xi_2 d\tilde{\eta}^3.$$

and take the result at a point of  $\Sigma$ , then we have, by (149), that  $\cos \alpha = 1$ ,  $\sin \alpha = 0$ ,  $d\alpha = 0$ . Also, by (147) and (148),

$$(154) \quad d\varphi \wedge \tilde{\eta}^3 = (\varphi_1 \eta^1 + \varphi_2 \eta^2 + \varphi_3 \eta^3) \wedge \tilde{\eta}^3 = \lambda_\omega \xi_1 \eta_1 \wedge \tilde{\eta}^3 + \lambda_\omega \xi_2 \eta_2 \wedge \tilde{\eta}^3,$$

hence, at points of  $\Sigma$ ,  $d\varphi \wedge \tilde{\eta}^3 = 0$ . In addition, from (126) it follows that at  $\Sigma$ ,  $d\tilde{\eta}^3 = 0$ . Therefore, on  $\Sigma$  we have

$$(155) \quad d\eta^1 = d\tilde{\eta}^1 - e^{-\varphi} d\xi_2 \wedge \tilde{\eta}^3 = d\tilde{\eta}^1 - d\xi_2 \wedge \eta^3.$$

From (150)–(152) it follows that on  $\Sigma$  we have

$$(156) \quad \tilde{\eta}^1 = \eta^1 + \xi_2 \eta^3$$

$$(157) \quad \tilde{\eta}^2 = \eta^2$$

$$(158) \quad \tilde{\eta}^3 = e^\varphi \eta^3,$$

Then, at points in  $\Sigma$  we have

$$(159) \quad d\tilde{\eta}^1 = \tilde{Q}_{23}^1 \tilde{\eta}^2 \wedge \tilde{\eta}^3 + \tilde{Q}_{31}^1 \tilde{\eta}^3 \wedge \tilde{\eta}^1 + \tilde{Q}_{12}^1 \tilde{\eta}^1 \wedge \tilde{\eta}^2 = (e^\varphi \tilde{Q}_{23}^1 - \xi_2 \tilde{Q}_{12}^1) \eta^2 \wedge \eta^3 + e^\varphi \tilde{Q}_{31}^1 \eta^3 \wedge \eta^1 + \tilde{Q}_{12}^1 \eta^1 \wedge \eta^2.$$

Set  $d\xi_2 = \xi_{21} \eta^1 + \xi_{22} \eta^2 + \xi_{23} \eta^3$ , and substitute it together with (159) to (155). Then we get

$$(160) \quad d\eta^1 = (e^\varphi \tilde{Q}_{23}^1 - \xi_2 \tilde{Q}_{12}^1 - \xi_{22}) \eta^2 \wedge \eta^3 + (e^\varphi \tilde{Q}_{31}^1 + \xi_{21}) \eta^3 \wedge \eta^1 + \tilde{Q}_{12}^1 \eta^1 \wedge \eta^2.$$

Thus,

$$(161) \quad Q_{23}^1 = e^\varphi \tilde{Q}_{23}^1 - \xi_2 \tilde{Q}_{12}^1 - \xi_{22}; \quad Q_{31}^1 = e^\varphi \tilde{Q}_{31}^1 + \xi_{21}; \quad Q_{12}^1 = \tilde{Q}_{12}^1.$$

In the same manner we prove that  $d\eta^2 = d\tilde{\eta}^2 + d\xi_1 \wedge \eta^3$ . By (142), we have

$$(162) \quad d\xi_1 = \mu d\lambda_\omega + \lambda_\omega d\mu = \mu(\lambda_2 \eta^2 + \lambda_3 \eta^3) + \lambda_\omega d\mu$$

By (128), we get that  $\lambda_3 = 0$  on  $\Sigma$ , so at points of  $\Sigma$  we have

$$(163) \quad d\xi_1 = \mu \lambda_2 \eta^2.$$

Then we get

$$(164) \quad Q_{23}^2 = e^\varphi \tilde{Q}_{23}^2 - \xi_2 \tilde{Q}_{12}^2 + \mu \lambda_2; \quad Q_{31}^2 = e^\varphi \tilde{Q}_{31}^2; \quad Q_{12}^2 = \tilde{Q}_{12}^2.$$

Thus we have proved

**Proposition 7.** *Let  $d\eta^a = C_{bc}^a \eta^b \wedge \eta^c$  be the structure equations of the adapted frame of the sub-Riemannian surface in a neighborhood of a point in  $\Sigma$ . The functions  $Q_{12}^1 = C_{12}^1|_\Sigma$  and  $Q_{12}^2 = C_{12}^2|_\Sigma$  do not depend on the choice of adapted frame, and so these functions are invariants of the surface.*

**Corollary 4.** *If  $V$  is an infinitesimal symmetry of the sub-Riemannian surface, then  $VQ_{12}^1 = 0$ , and  $VQ_{12}^2 = 0$ .*

*Proof.* Any symmetry  $f$  of the sub-Riemannian surface maps  $\Sigma$  onto itself, and sends a special form to a special form and the corresponding adapted frame to the corresponding adapted frame. Therefore,  $f^*Q_{12}^1 = Q_{12}^1$  and  $f^*Q_{12}^2 = Q_{12}^2$ . From this follows that an infinitesimal symmetry  $V$  is tangent to  $\Sigma$  and  $VQ_{12}^1 = VQ_{12}^2 = 0$ .  $\square$

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